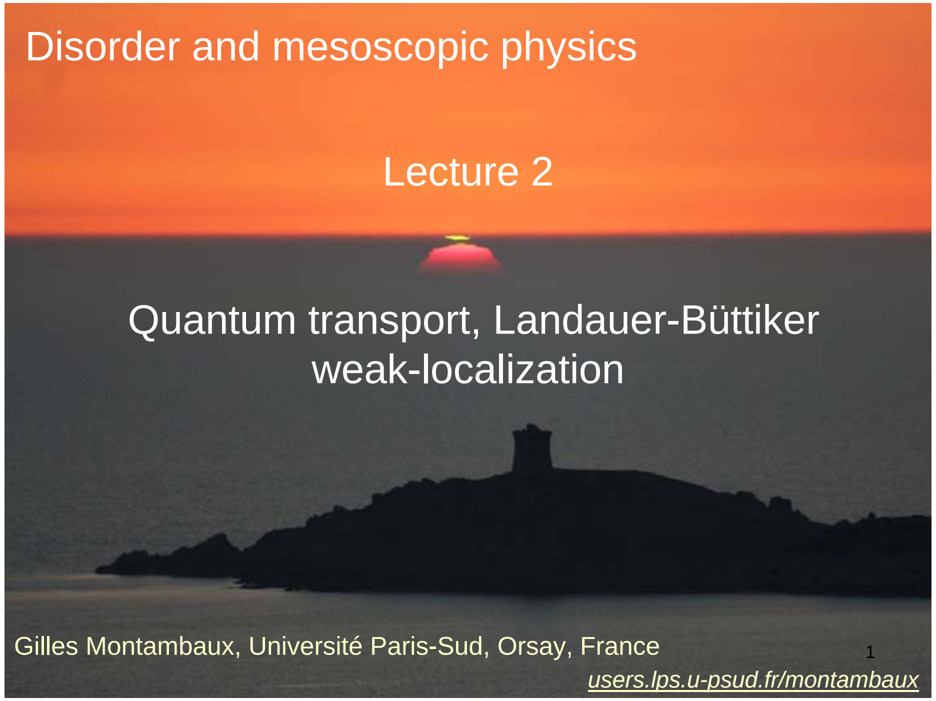


Disorder and mesoscopic physics

Lecture 2

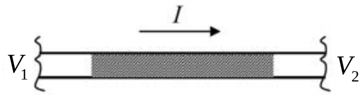
Quantum transport, Landauer-Büttiker weak-localization

Gilles Montambaux, Université Paris-Sud, Orsay, France

users.lps.u-psud.fr/montambaux

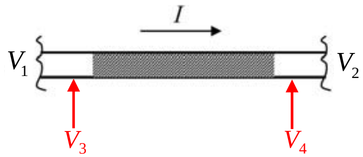
Landauer-Büttiker formulae

Two-terminal conductance



$$G_2 = 2\frac{e^2}{h}T$$

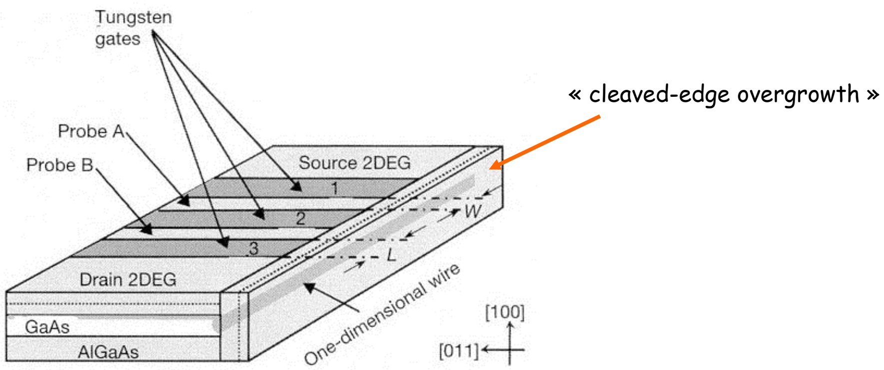
Four-terminal conductance



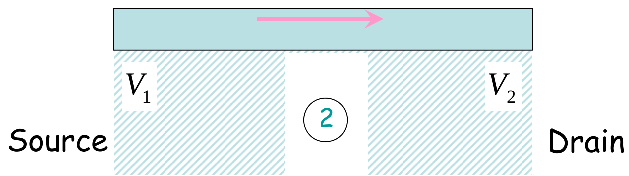
$$G_4 = 2\frac{e^2}{h}\frac{T}{R}$$

non-invasive voltage probes

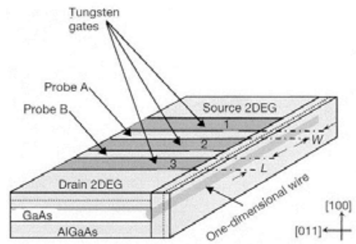
Four terminal resistance of a ballistic quantum wire (2001)



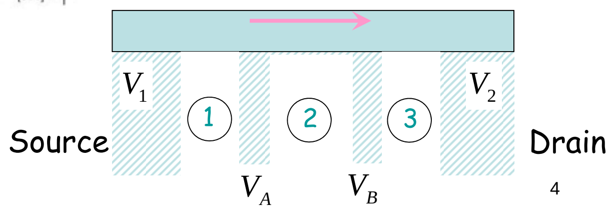
R. De Picciotto *et al.*, *Four terminal resistance of a ballistic quantum wire*, Nature 411, 51 (2001)

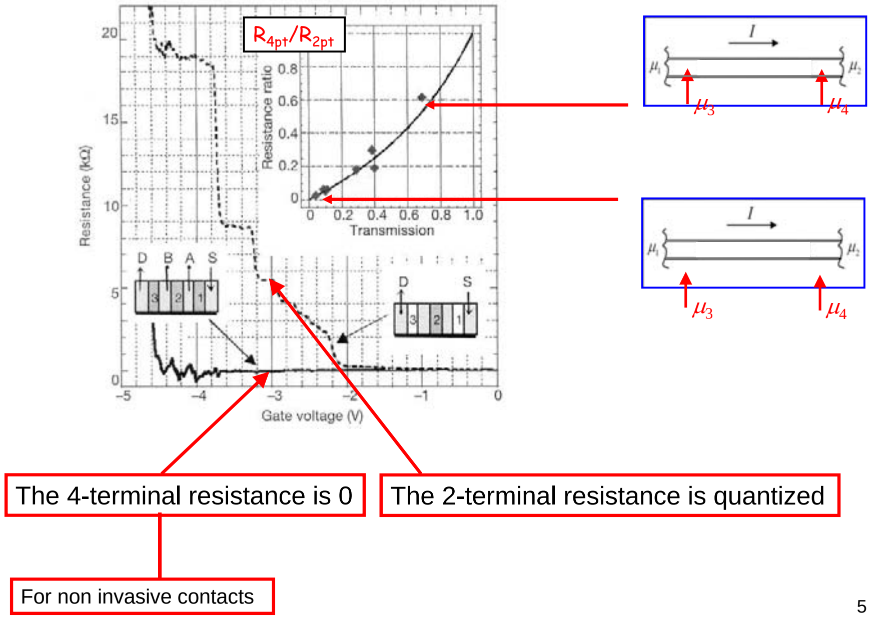


$$G_2 = 2\frac{e^2}{h}$$



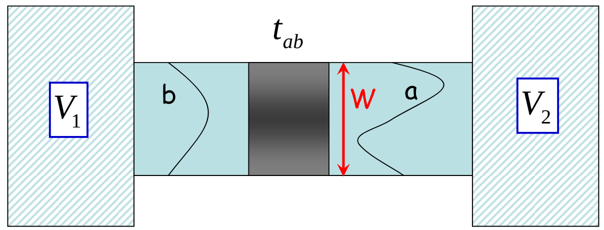
$$G_4 = \infty$$





R. De Picciotto *et al.*, *Four terminal resistance of a ballistic quantum wire*, Nature 411, 51 (2001)

Multichannel Landauer formula

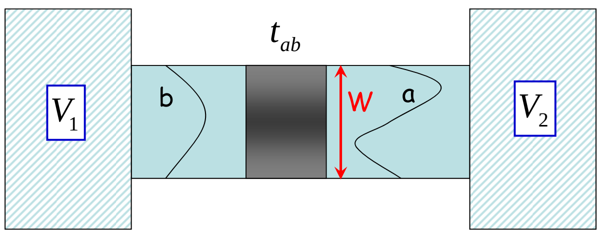


$$k_y = \frac{\pi}{W} \qquad\qquad k_y = \frac{2\pi}{W}$$

$$\lambda = 2W \qquad\qquad \lambda = W$$

The current is the sum of the contribution of the different | channels
| modes

Multichannel Landauer formula



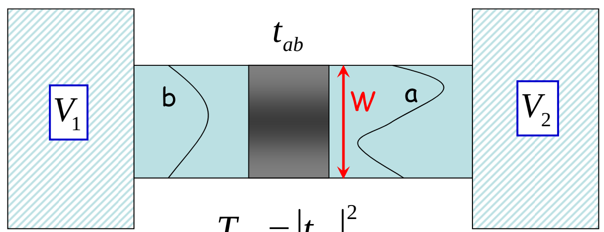
Current resulting from the transmission of a channel b to a channel a

$$I_{ab} = \frac{2e^2}{h} T_{ab} (V_1 - V_2) \qquad\qquad T_{ab} = |t_{ab}|^2$$

Total current

$$I = \frac{2e^2}{h} \sum_{a,b} T_{ab} (V_1 - V_2)$$

Multichannel Landauer formula

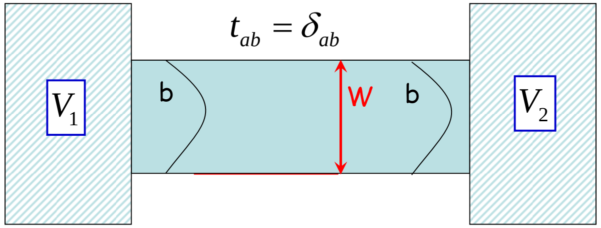


$$T_{ab} = |t_{ab}|^2$$

Multichannel Landauer formula

$$G = \frac{2e^2}{h} \sum_{a,b} T_{ab} \qquad\qquad G = \frac{2e^2}{h} \operatorname{tr} tt^\dagger$$

Multichannel Landauer formula : clean wave guide



$$T_{ab} = \delta_{ab}$$

$$G = \frac{2e^2}{h} \sum_{a,b} T_{ab} \quad \Rightarrow \quad G = \frac{2e^2}{h} M$$

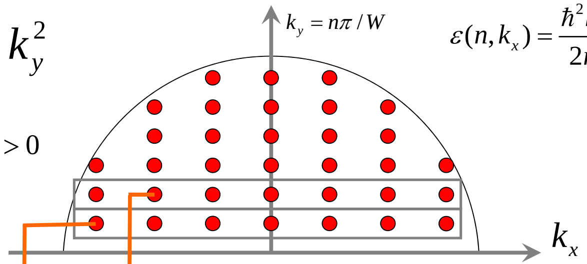
M is the number of transverse channels

Number of transverse channels

$$k_F^2 = k_x^2 + k_y^2$$

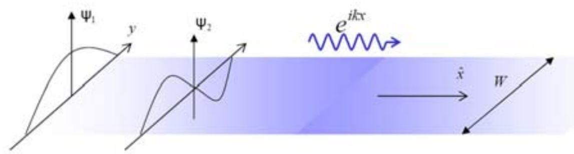
$$k_y = \frac{n_y \pi}{W}, \quad n_y > 0$$

$$\varepsilon(n, k_x) = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 n^2 \pi^2}{2mW^2}$$



$$k_y = \frac{\pi}{W} \qquad k_y = \frac{2\pi}{W}$$

$$M = \text{int}\left(\frac{k_F}{\pi / W}\right) = \text{int}\left(\frac{k_F W}{\pi}\right)$$



Number of transverse channels

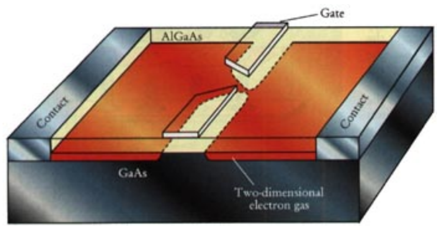
d=2 $\quad M = \text{int}\left(\frac{k_F}{\pi / W}\right) = \text{int}\left(\frac{k_F W}{\pi}\right)$

d=3 $\quad M = \text{int}\left(\frac{\pi k_F^2 / 4}{(\pi / W)^2}\right) = \text{int}\left(\frac{k_F^2 S}{4\pi}\right)$

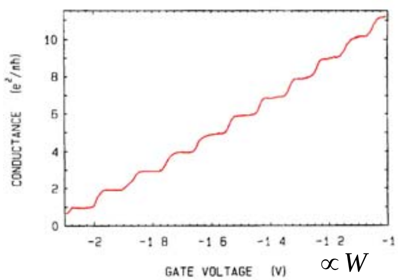
(d) $\quad M = \text{int}\left(\frac{A_{d-1} k_F^{d-1} / 2^{d-1}}{(\pi / W)^{d-1}}\right) = \text{int}\left(\frac{A_{d-1}}{(2\pi)^{d-1}} (k_F W)^{d-1}\right)$

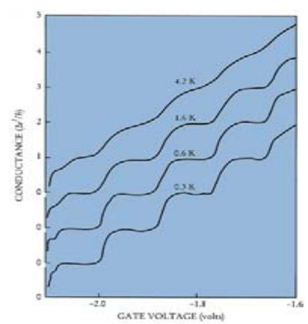
Quantization of the conductance (1988)

Quantum point contact **QPC**



$$G = 2\frac{e^2}{h} M = 2\frac{e^2}{h} \text{int}[\frac{k_F W}{\pi}]$$





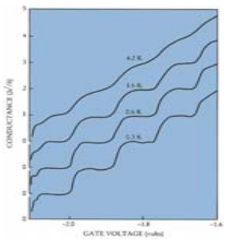
B.J van Wees et al., Quantized conductance of point contacts in a 2D electron gas, Phys. Rev. Lett. 60, 848 (1988)

Quantization of the conductance : temperature effect

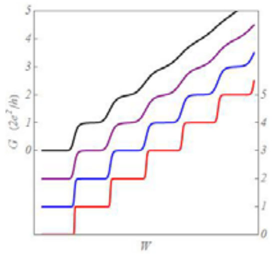
$$G = \frac{2e^2}{h}\sum_{a,b}\int T_{ab}(\varepsilon)\left(-\frac{\partial f}{\partial \varepsilon}\right)d\varepsilon$$

here : $$G = \frac{2e^2}{h}\int M(\varepsilon)\left(-\frac{\partial f}{\partial \varepsilon}\right)d\varepsilon$$

$$M(\varepsilon) = \sum \Theta(\varepsilon - \varepsilon_n)$$



$$G = \frac{2e^2}{h}\sum_n f(\varepsilon_n)$$



Characteristic energy : $$\varepsilon^* = \frac{\hbar^2\pi^2}{2m^* W^2}$$

$W \sim 250\,nm$

Quantization of the conductance : temperature effect

$$G = \frac{2e^2}{h}\sum_{a,b}\int T_{ab}(\varepsilon)\left(-\frac{\partial f}{\partial \varepsilon}\right)d\varepsilon$$

here : $$G = \frac{2e^2}{h}\int M(\varepsilon)\left(-\frac{\partial f}{\partial \varepsilon}\right)d\varepsilon$$

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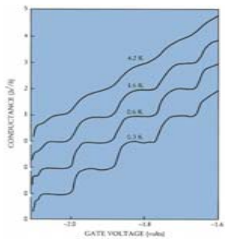
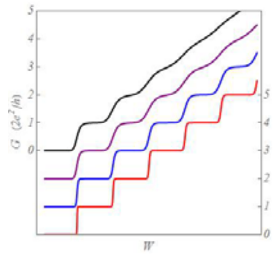


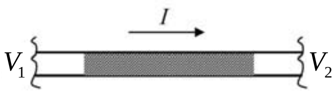
$$G = \frac{2e^2}{h}\sum_n f(\varepsilon_n)$$



Characteristic energy : $$\varepsilon^* = \frac{\hbar^2\pi^2}{2m^* W^2} \sim 1K$$

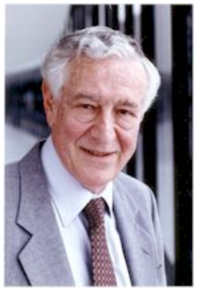
$W \sim 250\,nm$

Conductance = transmission

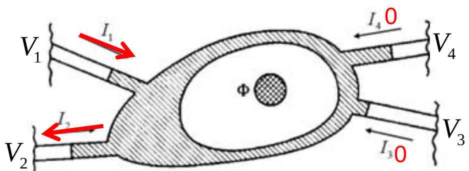


$$G = 2\frac{e^2}{h}T$$

Landauer formula



R. Landauer (1927-1999)

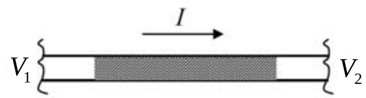


Landauer-Büttiker formalism

M. Büttiker (1950-2013)

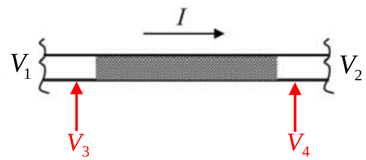
Landauer-Büttiker formulae

Two-terminal conductance

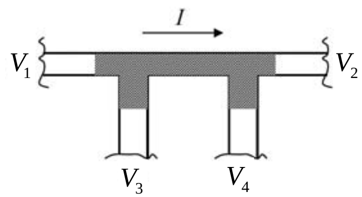


$$G_2 = 2\frac{e^2}{h}T$$

Four-terminal conductance



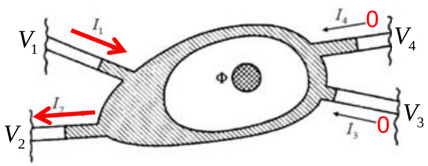
$$G_4 = 2\frac{e^2}{h}\frac{T}{R}$$



$$G_4 = 2\frac{e^2}{h}?$$

M. Büttiker

$$I_i = \frac{2e^2}{h}\left[(M_i - R_{ii})V_i - \sum_{j \neq i} T_{ij} V_j\right]$$



$$I = \bar{G} V$$

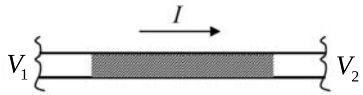
Conductance matrix

$$\bar{G} = \frac{2e^2}{h}\begin{pmatrix} M_1 - R_{11} & -T_{12} & -T_{13} & -T_{14} \\ -T_{21} & M_2 - R_{22} & -T_{23} & -T_{24} \\ -T_{31} & -T_{32} & M_3 - R_{33} & -T_{34} \\ -T_{41} & -T_{42} & -T_{43} & M_4 - R_{44} \end{pmatrix}$$

Four terminals

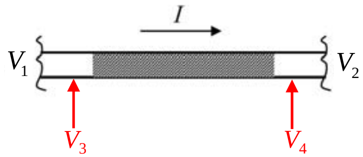
Landauer-Büttiker formulae

Two-terminal conductance

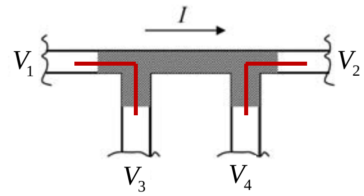


$$G_2 = 2\frac{e^2}{h}T$$

Four-terminal conductance



$$G_4 = 2\frac{e^2}{h}\frac{T}{R}$$

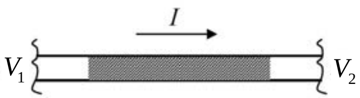


$$G_4 = 2\frac{e^2}{h}\frac{D}{T_{31}T_{42} - T_{32}T_{41}}$$

a priori depends on 9 transmission coefficients… (6 in zero field).

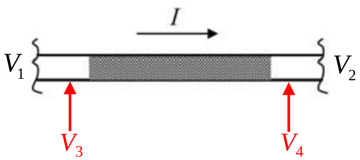
Landauer-Büttiker formulae

Two-terminal conductance

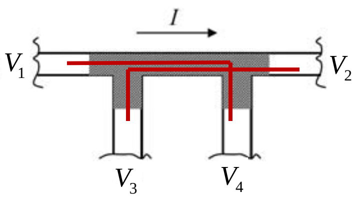


$$G_2 = 2\frac{e^2}{h}T$$

Four-terminal conductance



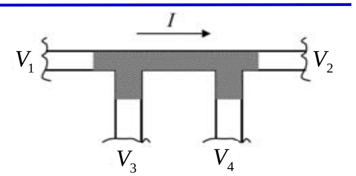
$$G_4 = 2\frac{e^2}{h}\frac{T}{R}$$



$$G_4 = 2\frac{e^2}{h}\frac{D}{T_{31}T_{42} - T_{32}T_{41}}$$

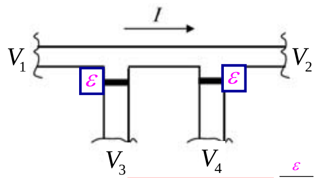
a priori depends on 9 transmission coefficients… (6 in zero field).

Landauer-Büttiker formulae

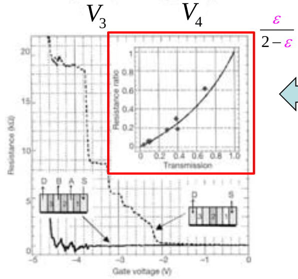


$$G_4 = 2\frac{e^2}{h}\frac{D}{T_{31}T_{42} - T_{32}T_{41}}$$

In de Picciotto experiment, 6 coefficients reduce to one

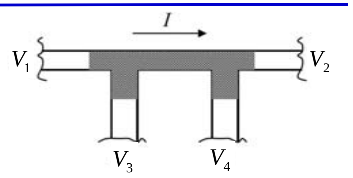


$$T_{31} = T_{42} = \varepsilon$$
$$T_{32} = T_{41} = \varepsilon\,(1-\varepsilon)$$
$$T_{34} = \varepsilon^2$$
$$T_{12} = (1-\varepsilon)^2$$

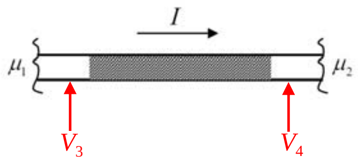


$$G_4(\varepsilon) = 2\frac{e^2}{h}\frac{2-\varepsilon}{\varepsilon}$$

Landauer-Büttiker formulae



Potential barrier + non invasive probe



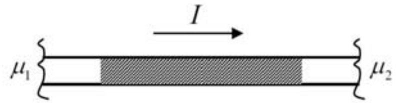
$$G_4 = 2\frac{e^2}{h}\frac{D}{T_{31}T_{42} - T_{32}T_{41}}$$

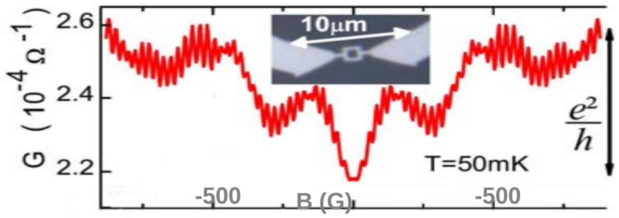
$$T_{31} = T_{42} = 0$$
$$T_{32} = T_{41} = 0$$
$$T_{34} = 0$$
$$T_{12} = T$$

$$G_4 = 2\frac{e^2}{h}\frac{T}{1-T}$$

Symmetry of the two-terminal conductance

$$T(B) = T(-B)$$

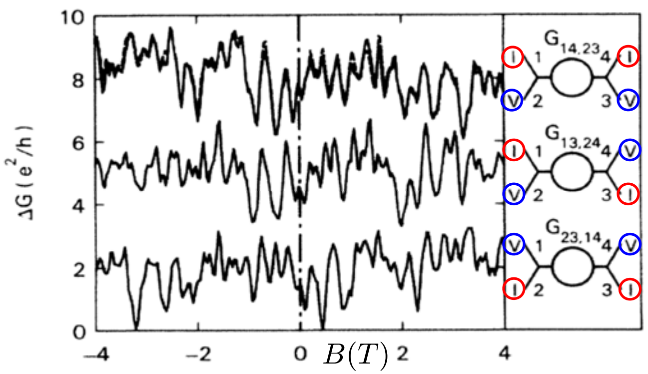




$$G(B) = G(-B)$$

L. Angers et al., Phys. Rev. B 75, 115309 (2007)

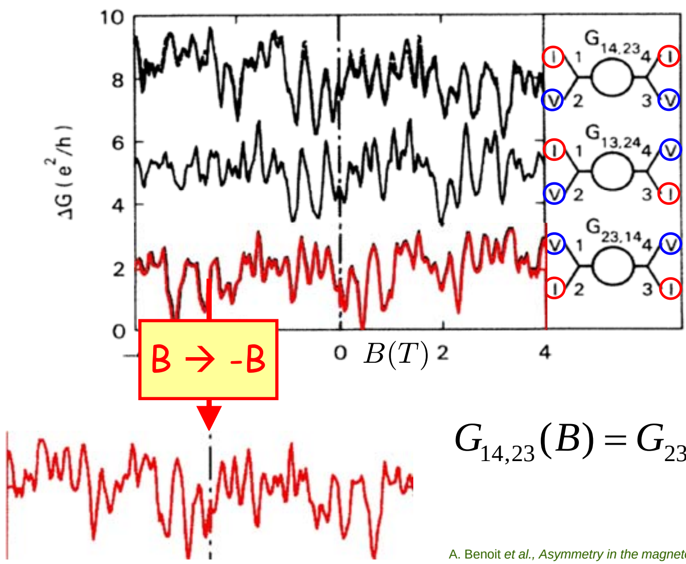
Symmetry of the four-terminal conductance ?



$$G_{12,34} = \frac{I_{12}}{V_{34}} = 2\frac{e^2}{h}\frac{D}{T_{31}T_{42} - T_{32}T_{41}}$$

A. Benoit *et al., Asymmetry in the magnetoconductance of metal wires and loops,* Phys. Rev. Lett. 57, 1765 (1986)

Symmetry of the four-terminal conductance ?



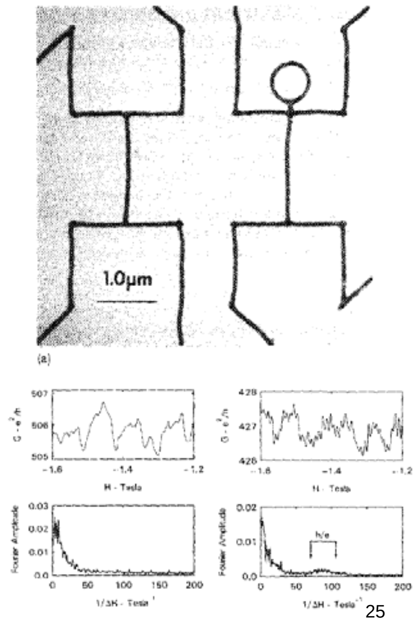
$$G_{14,23}(B) = G_{23,14}(-B)$$

A. Benoit *et al., Asymmetry in the magnetoconductance of metal wires and loops,* Phys. Rev. Lett. 57, 1765 (1986)

Phase coherence

⇓

Non- locality



Nonlocal electrical properties in mesoscopic devices

C. P. Umbach, P. Santhanam, C. van Haesendonck,[a] and R. A. Webb
IBM Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598

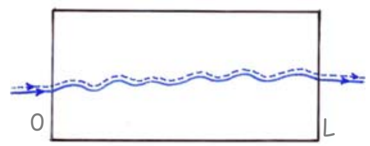
Appl. Phys. Lett. 50, 1289 (1987)

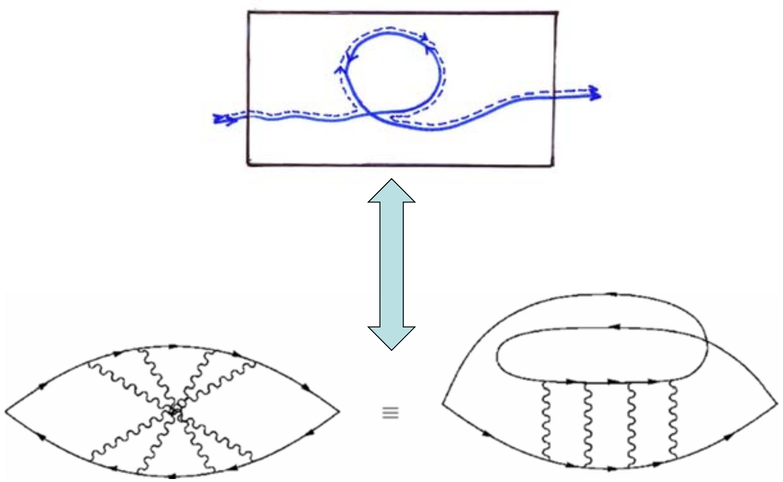
Conductance, transmission and probability

Conductance = transmission

Transmission through a disordered system= probability to cross the system

$$\overline{G} = 2\frac{e^2}{h}\overline{\sum_{a,b} T_{ab}} \propto P(0,L)$$





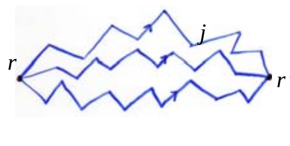
Diffusion probability, microscopic approach

$P(r,r',t)$ probability to find a particule at *r'* , if it has been injected at *r*

Quantum amplitude

Cf. Young's slits

$$G(r,r') = \sum_j A_j(r,r')$$



$$A_j(r,r') = |A_j(r,r')|\, e^{i\varphi_j(r,r')}$$

$$\varphi_j(r,r') = \frac{1}{\hbar}\int_r^{r'} p.dl$$

The probabiliy is the modulus square of the amplitude :

Disorder average

$$P(r,r') \sim \overline{|G(r,r')|^2} = \overline{\left|\sum_j A_j(r,r')\right|^2}$$

Diffusion probability, microscopic approach

Two contributions

Disorder average

$$P(r,r') = \overline{\sum_j |A_j(r,r')|^2} + \overline{\sum_{j\neq j'} A_j(r,r')A^*_{j'}(r,r')}$$

Classical term — Interference term

⟶ Quantum effects

Classical transport : only paired trajectories A_j A_j contribute
If the trajectories are different, the amplitudes A_j et $A_{j'}$ are different

⟹ uncorrelated phases
⟹ In average, the interference term disappears

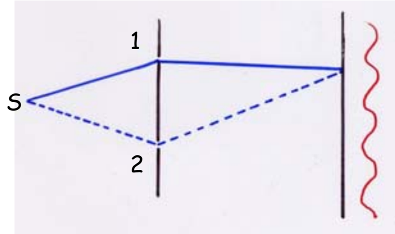
$P(r,r')$ =

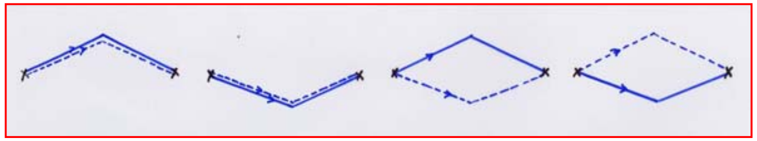
$P_{cl}(r,r')$

Example : Young slits

$I = I_{cl} + I_{\text{int}}$

$$I = |A_1 + A_2|^2$$

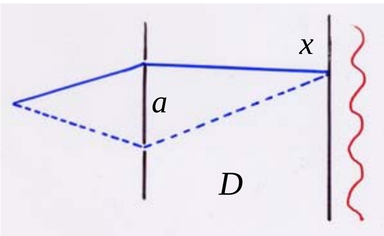




$$I = |A_1|^2 + |A_2|^2 + A_1\,A_2^* + A_2\,A_1^*$$

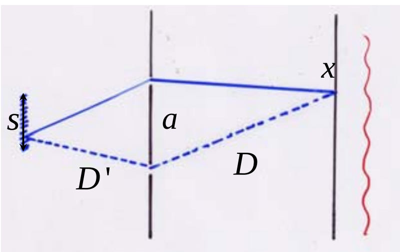
Example : Young slits

$I = I_{cl} + I_{\text{int}}$



$$I = I_{cl}\left(1+\cos\frac{kax}{D}\right)$$

$$I = I_{cl} + I_{\text{int}}$$



$$I = I_{cl}\left(1+\frac{\sin\frac{kas}{D'}}{\frac{kas}{D'}}\cos\frac{kax}{D}\right)$$

$$\langle I_{\text{int}}\rangle = 0$$

Diffusion probability, microscopic approach

Two contributions

$$P(r,r') = \overline{\sum_j |A_j(r,r')|^2} + \overline{\sum_{j\neq j'} A_j(r,r')A^*_{j'}(r,r')}$$

Classical term — Interference term

⟶ Quantum effects

Classical transport : only paired trajectories A_j A_j contribute
If the trajectories are different, the amplitudes A_j et $A_{j'}$ are different

⟹ uncorrelated phases
⟹ In average, the interference term disappears

$P(r,r')$ =

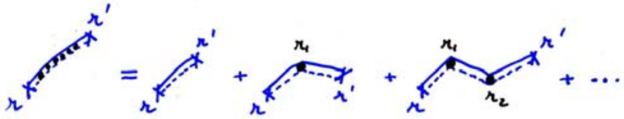
$P_{cl}(r,r')$

DIFFUSON

$P_{cl}(r,r')$

Classical probability, microscopic approach

$P(r,r')$ Probability to go from r to r'



$$P(r,r') = P_0(r,r') + P_0(r,r_1)\bullet P_0(r_1,r') + P_0(r,r_1)\bullet P_0(r_1,r_2)\bullet P_0(r_2,r') + \ldots$$

= Probability to go from r to r' without any collision
+ probability to go from r to r' with one collision + etc.

→ Iterative structure (Bethe-Salpeter equation)

$$P(r,r') = P_0(r,r') + P_0(r,r_1)\bullet P(r_1,r')$$

« building block »

Classical probability, microscopic approach

Iteration in Fourier space :

$$P(q,\omega) = \frac{P_0(q,\omega)}{1 - P_0(q,\omega)/\tau_e}$$

« building block »

$$P_0(R,t) = \frac{\delta(R - vt)}{4\pi R^2}\ e^{-t/\tau_e} \qquad d=3$$

Diffusive limit $\omega\tau_e \ll 1$ $ql_e \ll 1$

$$D = \frac{v^2\tau_e}{d} = \frac{v\,l_e}{d}$$

$$P_0(q,\omega) \sim \tau_e(1 + i\omega\tau_e - Dq^2\tau_e)$$

$$\left(-i\omega + Dq^2\right)P_{cl}(q,\omega) = 1$$

$$P_0(\vec{q},\omega) = \int \frac{\delta(R - vt)}{4\pi R^2} e^{-t/\tau_e}\ e^{i\omega t - iqR\cos\theta}\ 2\pi R^2 \sin\theta dR d\theta dt$$

$$P_0(\vec{q},\omega) = \frac{1}{v}\int e^{-R/l_e + i\omega R/v - iqR\cos\theta}\ 2\pi \sin\theta \frac{d\theta}{4\pi} dR$$

$$P_0(\vec{q},\omega) = \frac{1}{v}\left\langle \frac{1}{1/l_e - i\omega/v + iq\cos\theta}\right\rangle_\theta$$

$$P_0(\vec{q},\omega) = \left\langle \frac{\tau_e}{1 - i\omega\tau_e + iql_e\cos\theta}\right\rangle_\theta \qquad \langle q^2 l_e^2 \cos^2\theta\rangle = \frac{q^2 l_e^2}{d} = Dq^2\tau_e$$

Diffusive limit $\omega\tau_e \ll 1$ $ql_e \ll 1$

$$P_0(\vec{q},\omega) = \tau_e(1 + i\omega\tau_e - Dq^2\tau_e)$$

Classical probability, diffusion equation

$P_{cl}(r,r')$ is solution of a classical diffusion equation:

$$\left(\frac{\partial}{\partial t} - D\Delta\right)P_{cl}(r,r',t) = \delta(r - r')\delta(t) \qquad P_{cl}(q,t) = e^{-Dq^2 t}$$

Solution in free space in d dimensions: $P_{cl}(R,t) = \frac{1}{(4\pi Dt)^{d/2}} e^{-\frac{R^2}{4Dt}}$

An important result, the return probability: $P(r,r,t) = \frac{1}{(4\pi Dt)^{d/2}}$

Typical distance: $\langle R^2(t)\rangle = 2d\ Dt$

Remarks about classical diffusion : Thouless and recurrence times

Typical time to reach the edge of the system

Diffusion time
Thouless time

$$\tau_D = \frac{L^2}{D}$$

Total time spent near the origin in a finite system : recurrence time

Remarks about classical diffusion : Thouless and recurrence times

Probability is normalized such as $\int P(\boldsymbol{r},\boldsymbol{r}',t)d\boldsymbol{r}' = 1$

Probability to be in a given volume V after time t $\int_V P(\boldsymbol{r},\boldsymbol{r}',t)d\boldsymbol{r}'$

Total time spent in a given volume V after time t $T = \int_0^t dt \int_V P(\boldsymbol{r},\boldsymbol{r}',t)d\boldsymbol{r}'$

Total time spent near the origin $\tau_R = v\int_0^\infty dt P(\boldsymbol{r},\boldsymbol{r},t)$

In a finite system : $\tau_R = \int_{\tau_e}^{\tau_D} \frac{v}{(4\pi D t)^{\frac{d}{2}}} dt$

Remarks about classical diffusion : recurrence time

Typical time to reach the edge of the system

Diffusion time
Thouless time

$$\tau_D = \frac{L^2}{D}$$

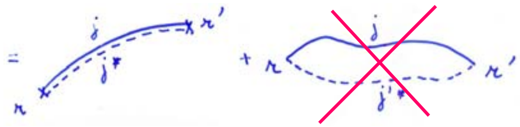
Total time spent near the origin in a finite system : recurrence time

$$\tau_R = \int_{\tau_e}^{\tau_D} \frac{v}{(4\pi D t)^{\frac{d}{2}}} dt$$

$d=1$	$\frac{\tau_R}{\tau_e} \propto$	$\frac{L}{l_e}$
$d=2$	$\frac{\tau_R}{\tau_e} \propto$	$\ln\frac{L}{l_e}$
$d=3$	$\frac{\tau_R}{\tau_e} \propto$	1

Diffusion probability, quantum corrections

$P(r,r')$ =



$P_{cl}(r,r')$

?

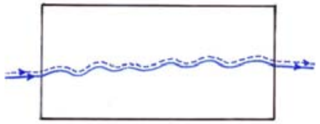
$$P(r,r') = \sum_j \left|A_j(r,r')\right|^2 + \sum_{j\neq j'} A_j(r,r')A^*_{j'}(r,r')$$

Conductance, quantum corrections

The conductance is proportional to the probability to transfer electrons from one side of the sample to the other side (Landauer-Buttiker)

$$\overline{G} \propto P(0,L)$$

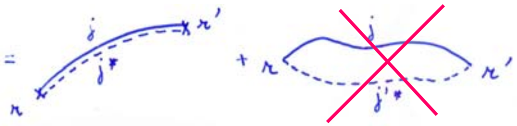
Classical condutance :



$$\overline{G}_{cl} \propto P_{cl}(0,L)$$

Beyond the classical contribution ?

$P(r,r') =$



$$P_{cl}(r,r')$$

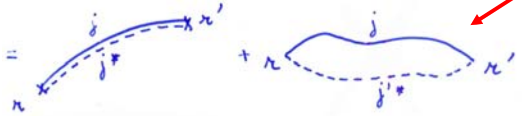
DIFFUSON

+ quantum corrections

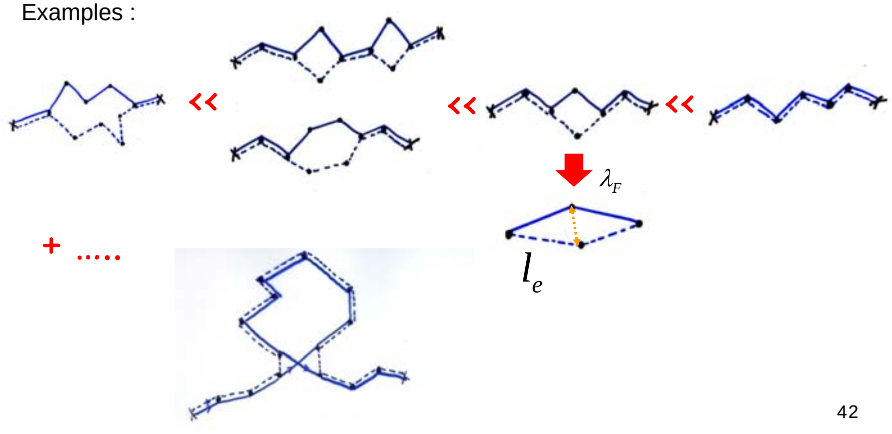
Quantum corrections

$$P(r,r') = \overline{\sum_j |A_j(r,r')|^2} + \overline{\sum_{j\neq j'} A_j(r,r')A^*_{j'}(r,r')}$$

$P(r,r') =$



Examples :



Quantum corrections

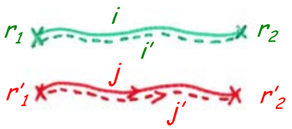
$= P_{cl}(r,r')$

Q: when do quantum effects appear ?

A : When trajectories cross

Example : 2 particules from r_1,r_2 *to* r_1',r_2'

$$\mathcal{P}(r_1,r_2,r_1',r_2') = \overline{G(r_1,r_2)G^*(r_1,r_2)G(r_1',r_2')G^*(r_1',r_2')}$$

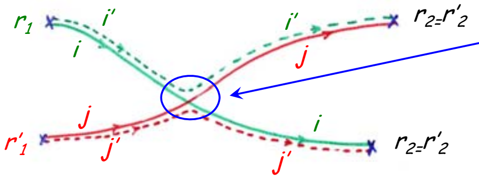


$$= \overline{\sum_{ii'jj'} A_i(r_1,r_2)A^*_{i'}(r_1,r_2)A_j(r_1',r_2')A^*_{j'}(r_1',r_2')}$$

Classically : product of probabilities

$$\mathcal{P}(r_1,r_2,r_1',r_2') = P_{cl}(r_1,r_2)P_{cl}(r_1',r_2')$$

Quantum correction :

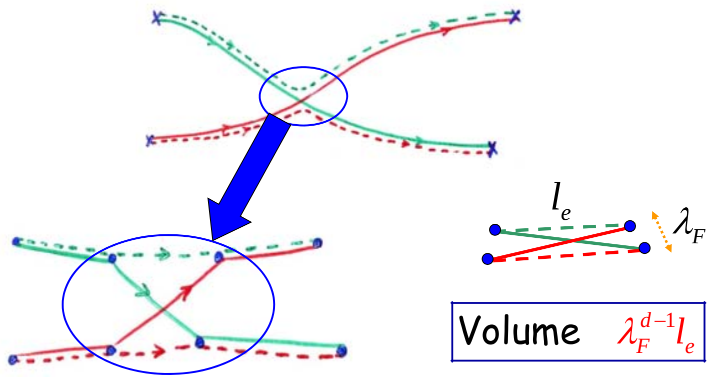


Quantum crossing
= exchange of amplitudes

« Hikami box »

if r_2=r'_2

Quantum crossing



Simple picture : the diffuson $P_{cl}(r,r',t)$ is an object of length $v_F t$ and cross-section λ_F^{d-1}

* Quantum effects are due to crossings

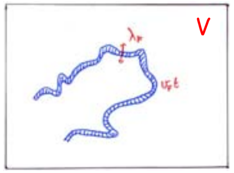
* Importance of quantum effects => evaluate the probability of crossing

Evaluation of the probability of quantum crossing

Simple picture : the diffuson $P_{cl}(r,r',t)$ is an object of length $v_F t$ and cross-section λ_F^{d-1}

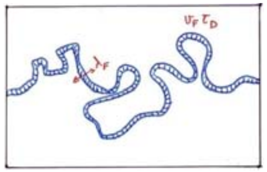
Probability of crossing during a time t, in a volume $V=L^d$

$$p_\times(t) = \frac{\lambda_F^{d-1} v_F t}{L^d}$$



Transport phenomena : in a sample of size L, the wave spends a time $\tau_D=L^2/D$
The probability of quantum crossings which affects tranport properties is thus

$$p_\times(\tau_D) = \frac{\lambda_F^{d-1} v_F \tau_D}{L^d} \sim \frac{1}{g} \quad !!!$$



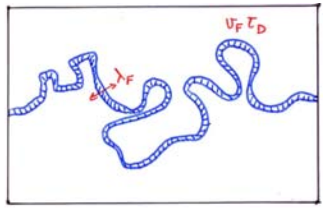
g Dimensionless conductance ! $g = G/(2e^2/h)$

Conductance : ratio of two volumes

$$g \sim \frac{V}{\lambda_F^{d-1} v_F \tau_D}$$

Volume of the system

Volume of a tube of section λ_F which diffuses across the system



Coherent effects and quantum crossings

The probability of quantum crossings is $1/g$

Quantum crossings → Quantum corrections
Quantum crossings → Fluctuations, correlations

Classical transport $G_{cl} = g\,\dfrac{2e^2}{h}$

Quantum effects are of order
(fluctuations, oscillations, corrections)

$$G_{cl} \times \frac{1}{g} \sim \boxed{\frac{e^2}{h}}$$

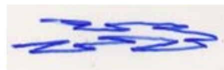
example: Aharonov-Bohm (or Sharvin-Sharvin) oscillations in a metal are of order e^2/h

In a good metal ($g \gg 1$), quantum effects are small

1D versus quasi-1D

one channel: strictly 1D motion

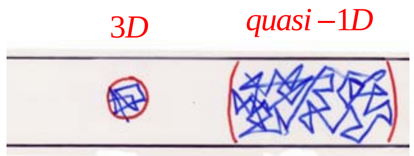
$\lambda_F > W$



1D wire

many channels, 3D motion, but 1D diffusion

$\lambda_F < W$



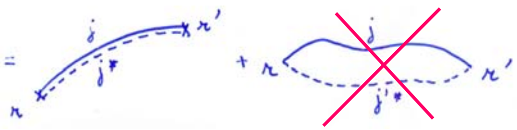
Quasi-1D wire

Number of channels : $M = \dfrac{(k_F W)^2}{4\pi}$

Summary of previous lecture

conductance ~ transmission ~ probability

$P_{cl}(r,r') =$



classical diffuson quantum corrections

quantum crossing → 1 / g correction

$$p_\times(\tau_D) \sim \frac{\lambda_F^{d-1} v_F \tau_D}{V} \sim \frac{1}{g}$$

classical transport $\propto g\frac{e^2}{h}$

quantum effects $\propto \frac{e^2}{h}$

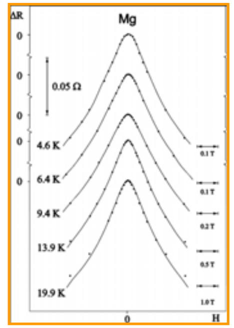
Weak localization $\overline{G} = G_{cl} + \delta G(B)$

Variation of the resistance R(B) of a metallic film
as a function of the applied magnetic field

The magnetic field scale depends on temperature

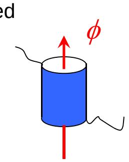
A coherent contribution, deviation to Ohm's law, increases
the resistance and is destroyed by a magnetic field:

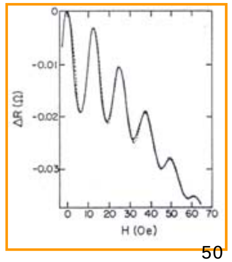
Negative magnetoresistance : signature of weak localization



magnetoresistance of Mg film (Bergmann 1984)

In a cylinder, the resistance is periodically modulated
with the flux through the cylinder,
with a period $\phi_0/2$.





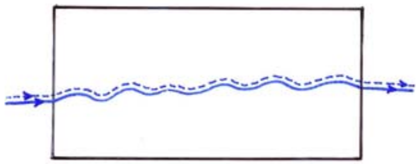
« Sharvin-Sharvin » oscillations (1981)

A phase coherence effect which resists disorder

Weak localization

Classical conductance

G_{cl}

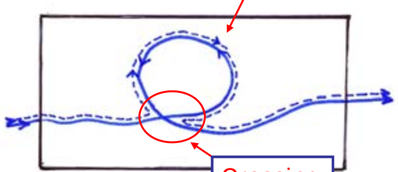


$\propto P_{cl}(0,L)$

Quantum correction ⇨ **One crossing** ⇨ **One loop**

$$\frac{\Delta G}{G_{cl}} \sim -\frac{1}{g}\langle P_{\text{int}}(t)\rangle$$

$$\Delta G \sim -\frac{2e^2}{h}\langle P_{\text{int}}(t)\rangle$$

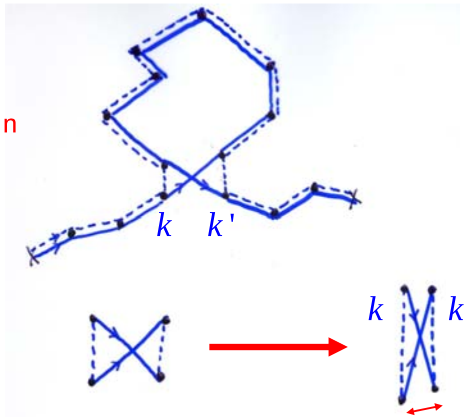


$\propto \Delta P(0,L)$

$P_{\text{int}}(t)$ = distribution of number of loops with time t = return probabililty

Weak localization : (-) sign

(-) sign



l_e

λ_F

$k' \sim -k$

backscattering

Weak-Localization

Loops and quantum crossings
Nb loops, probability $P_{int}(r,r',t)$, return probability
Magnetic field, phase coherence
Weak-localisation in dimension d

A few solutions of the diffusion equation and WL
Magnetic field and negative magnetoresistance
Magnetic field in quasi-1D wires
AAS oscillations