

Symmetry of electrical conduction

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The resistance of a conductor measured in a four-probe setup is invariant if the exchange of the voltage and current sources is accompanied by a magnetic field reversal. We present a derivation of this theorem. The reciprocity of the resistances is linked directly to the microscopic reciprocity of the S-matrix, which describes reflection at the sample and transmission through the sample. We demonstrate that this symmetry holds for a conductor with an arbitrary number of leads. Since leads act like inelastic scatterers, consideration of a many-probe conductor also implies that the reciprocity of resistances is valid in the presence of inelastic scattering. Various conductance formulae are discussed in the light of the reciprocity theorem. Finally, we discuss some implications of our results for the nature of a voltage measurement and point to the difference between chemical potentials and the local electric potential.

1. Introduction

Symmetries are of paramount importance, since they force certain constraints on the laws of physics. Once established, symmetries can provide sample tests of experimental accuracy and greatly reduce the amount of data which has to be taken. In this paper we are concerned with the reciprocity

theorem for electrical conductors. This theorem applies to conductors which are connected to several contacts. In a four-probe experimental setup, two of these contacts are used to supply and draw current from the sample, and the potential difference is measured between another pair of probes. The reciprocity theorem, in the absence of a magnetic field, states that the resistance measured in a particular configuration of current and voltage leads is equal to the resistance in the configuration where the current and voltage leads have been exchanged,

$$\mathcal{R}_{mn,kl} = \mathcal{R}_{kl,mn}. \quad (1)$$

Here the first pair of indices represents the contacts used to supply and draw current, and the second pair of indices represents the probes used to measure the potential difference. Reciprocity of current sources and voltage sources has long been understood. Searle [1], in his 1911 article, presents a derivation of Equation (1) which he attributes to Heaviside. A more recent discussion is given by van der Pauw [2]. The reciprocity theorem is rarely mentioned in modern textbooks, which instead emphasize the Onsager-Casimir symmetry relations of the *local* conductivity tensor [3, 4],

$$\sigma_{\alpha\beta}(H) = \sigma_{\beta\alpha}(-H). \quad (2)$$

Here the indices refer to coordinates and not, as in Equation (1), to the contacts. The extension of Equation (1) to the case of a magnetic field, surprisingly, is of more recent origin. In the presence of a magnetic field the exchange of the current and voltage leads has to be accompanied by a reversal of the magnetic field,

$$\mathcal{R}_{12,34}(H) = \mathcal{R}_{34,12}(-H). \quad (3)$$

The reciprocity theorem, Equation (3), is related to the

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Onsager–Casimir symmetry relations, Equation (2); Spal [5] and Sample et al. [6] give a derivation of Equation (3) which rests on the validity of the local symmetry relations, Equation (2). Thus, by using Equation (2) one can arrive at Equation (3). On the other hand, verification of Equation (3) in a particular four-probe geometry does not imply the validity of Equation (2). In fact, according to Casimir [4], a whole series of four-probe measurements with differing geometrical arrangement of the contacts must yield equivalent results for Equation (2) to be valid. It is clear, therefore, that conductors might exist which do not obey the local Onsager–Casimir symmetry relations, Equation (2), but which nevertheless obey the *global* symmetry, Equation (3). Hence, it is desirable to derive Equation (3) directly without invoking Equation (2). A direct derivation of the reciprocity theorem, Equation (3), was given by the author in [7] and is reviewed and extended in this paper.

Our interest in these symmetries stems from a concern for electron conduction in tiny disordered conductors. We refer the reader to some review papers and papers with a large number of citations on this topic [8–16]. Interesting effects in such small conductors arise from the quantum-mechanical nature of electron transport. If the wave-like nature of the carriers plays a role, the relation between the current and the electric field cannot be local. The symmetry properties of the magnetoresistance have been of interest to us for some time. Early experiments [9, 17] in quest of h/e oscillations in disordered normal loops [18–23] revealed a magnetoresistance which was not symmetric with regard to field reversal. In view of the prevailing (and mistaken) expectation at that time that these experiments were designed to measure a longitudinal conductance, this was noticeable and triggered our attention. A possible explanation of this asymmetry was offered by the observation [24] that a conductance formula due to Azbel [21, 25] is also not symmetric under field reversal. However, as it turned out, the asymmetry given by this conductance formula is, for metallic conductors in the diffusive regime, too small to account for the experimentally observed effect. Others argued that the asymmetry was not an intrinsic property but could be due to magnetic impurities [26]. It was in conjunction with an additional experiment, carried out by Benoit et al. [27], specifically designed to clarify the nature of the asymmetry, that we derived a resistance formula [7] for quantum coherent electron transport which also obeys the reciprocity symmetry given by Equation (3). The resistances obtained in [7] are related to the probabilities of carriers for transmission through the sample and reflection at the sample. The possibility of relating the resistance of a sample directly to transmission and reflection probabilities was pointed out by Landauer [28, 29]. The sample is viewed as a target at which carriers are reflected or transmitted. In contrast to the Greenwood–Kubo formulation, the resistance is related to *static* scattering

properties of the sample. The point of view advanced in [28] and [29] has received increasing attention since the beginning of this decade [21, 24, 25, 30–35]. However, despite the fact that four-terminal conductors have been studied [32], and other results [21, 25] have been interpreted as four-terminal resistances [34, 35], a resistance formula with the symmetry of Equation (3) was lacking. The derivation of Equation (3) given in [7] assumed a disordered sample which scatters carriers only elastically. Inelastic scattering is assumed to be spatially separated [28, 29] from the conductor, and occurs only in the reservoirs (see Figure 1, shown later). Instead of a uniform magnetic field which penetrates the conductor and reaches the carriers, a field is introduced via an Aharonov–Bohm flux through a hole in the conductor. The reciprocity theorem was derived by first demonstrating the global Onsager–Casimir symmetry relations for the conductances relating the currents in the leads to the chemical potentials of the reservoirs. Reference [7] relates the reciprocity relations of the resistances directly to the reciprocity of the scattering matrix, the *S*-matrix, describing transmission and reflection of carriers at the sample. A basic feature of the resistance formula derived in [7] is the equivalent quantum-mechanical treatment of the contacts which are used to carry current to and from the sample and those which are used to measure voltages. Previous works on conductance formulae have made a number of either implicit or explicit assumptions on what constitutes a voltage measurement. Our work implies that these assumptions have to be revised, and in Sections 3 and 5 we briefly return to this subject.

In submicron structures, at very low temperatures, the magnetoresistance is sensitive to the specific configuration of the impurities and inhomogeneities. Thus, the experiment of Benoit et al. [27] must be regarded as a particularly sensitive test of the symmetries predicted by Equation (3). Even in larger samples the symmetries predicted by Equation (3) have been tested only recently. In connection with the von Klitzing effect, the reciprocity relations have been experimentally confirmed by Sample et al. [6]. High-field magnetoresistance measurements on single-metal samples of a shape with low symmetry have been carried out by Soethout et al. [36], who find generally good agreement with Equation (3) except for small discrepancies which they attribute to the possibility of structural changes induced by Hall currents. In view of these experiments, it is clear that the reciprocity theorem is a fundamental physical law.

In this paper we extend the derivation of Equation (3) given in [7]. We show that the validity of Equation (3) is independent of the number of leads attached to the sample. Since leads leading away from a conductor to a reservoir, in which carriers suffer phase-randomizing events, act like inelastic scatterers [37–40], this demonstrates that, as expected, the symmetries of Equation (3) apply to conductors which are large compared to an inelastic length.

This then also removes the need to keep the reservoirs (the measuring pads) close to the sample. Since the reciprocity theorem is fundamental, it is worthwhile discussing the symmetry of other expressions derived for the conductance. We analyze the four-terminal interpretation of the Landauer formula [28, 29], which yields a resistance proportional to $R/(1 - R)$, and the many-channel generalization of this expression [21, 25, 34] in the light of the reciprocity theorem. We point to certain interference terms which are neglected in these conductance formulae.

Recent experiments by Benoit et al. [41] and Skocpol et al. [42] further demonstrate that the probes are an integral part of the conductor. In these experiments the voltage difference is measured on leads which are separated by less than a phase-breaking length. To understand these experiments, it is essential to take into account that carriers can make large excursions into the voltage probe and experience inelastic events in such a probe [40]. A diagrammatic discussion of these phenomena has been put forth by Maekawa et al. [43], Kane et al. [44, 45], and Hershfield and Ambegaokar [46]. References [44] and [45] investigate the connection of the results of [7], expressing resistances in terms of transmission probabilities, with the Greenwood-Kubo linear response formalism. An alternative way to calculate transmission probabilities is by direct computation [47, 48], and for multiport conductors this has been achieved by Baranger et al. [49, 50]. We do not address the statistical aspects of voltage fluctuations; instead, we focus on the implications of [7] for the definition of resistance and voltage measurement.

Before concluding this section we mention, for completeness, a further generalization of the reciprocity theorem. Deviations from Equation (3) can occur if the sample admits a magnetic moment M . In such a case, as pointed out by Strikhsman and Thomas [51], the conductivity tensor obeys $\sigma_{\alpha\beta}(H, M) = \sigma_{\beta\alpha}(-H, -M)$. Correspondingly, the reciprocity theorem for a conductor with a magnetization M can be stated as

$$\mathcal{R}_{mn,kl}(H, M) = \mathcal{R}_{kl,mn}(-H, -M); \quad (4)$$

i.e., the exchange of leads must be accompanied by a reversal of the magnetic field H and the magnetization M .

2. Multiprobe conductance formula

Consider the conductor shown in **Figure 1**. A field dependence is introduced by studying the response of the conductor to an Aharonov-Bohm flux through the hole [18-21]. In a uniform magnetic field, there are, in addition to the resistance oscillations with fundamental period [18-23] $\Phi_0 = hc/e$, also aperiodic resistance variations as a function of the magnetic field [9, 17, 48]. While we focus on the Aharonov-Bohm oscillations, our conclusions apply equally to the aperiodic resistance variations [40-46, 48-50]. In the presence of a uniform magnetic field, a two-

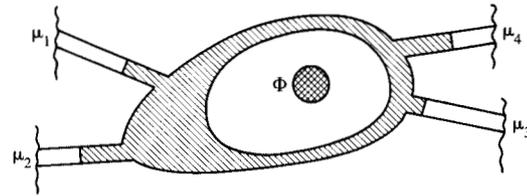


Figure 1

Disordered normal conductor with four terminals connected via perfect leads (unshaded) to four reservoirs at chemical potentials, μ_1 , μ_2 , μ_3 , and μ_4 . An Aharonov-Bohm flux Φ is applied through the hole of the sample. From [7], reprinted with permission.

dimensional disordered conductor can be considered as a network of microscopic loops of size a with a flux $\Phi = Ha^2 \ll \Phi_0 = hc/e$ threading each loop. The leads in Figure 1 are connected to reservoirs which are at chemical potentials μ_1 , μ_2 , μ_3 , μ_4 , respectively. The reservoirs serve both as a source and as a sink of carriers and have the following properties: At zero temperature they feed the leads with carriers up to the energy μ_i . Every carrier coming from the lead and reaching the reservoir is absorbed by the reservoir irrespective of the phase and energy of the incident carrier. Technically, it is convenient to introduce a piece of perfect wire (unshaded part of the leads in Figure 1), free of elastic scattering, between the disordered terminals and the reservoirs. First we assume that these perfect leads are strictly one-dimensional quantum channels; i.e., there are only two states at the Fermi energy, one with positive velocity (taken to be the direction away from the reservoir) and one with negative velocity. The multichannel case is discussed later. Scattering in the sample is elastic; inelastic events occur only in the reservoirs. The elastic scattering properties of the sample are described by an S-matrix, which relates the amplitudes α'_i , $i = 1, \dots, 4$, of the outgoing currents to the amplitudes α_i of the incident currents,

$$\alpha'_i = \sum_{j=1}^{i=4} s_{ij} \alpha_j.$$

Since current is conserved, the S-matrix is unitary, $S^+ = S^{-1}$. Here $+$ denotes Hermitian conjugation. Time reversal implies $S^*(-\Phi) = S^{-1}(\Phi)$. Here the star denotes complex conjugation. Hence, the S-matrix obeys the reciprocity relations $s_{ij}(\Phi) = s_{ji}(-\Phi)$. The transmission amplitude $s_{ij}(\Phi)$ for a carrier incident in contact j to reach contact i in the presence of a flux Φ is the same as that of a carrier incident in contact i to reach contact j if the flux has been

reversed. Below we use the reciprocity of the S-matrix to derive the reciprocity theorem for the electrical resistance. We only invoke the probabilities $T_{ij} = |s_{ij}|^2$, $i \neq j$ for transmission of carriers incident in lead j to reach lead i and the probabilities $R_{ii} = |s_{ii}|^2$ for carriers incident in lead i to be reflected into lead i . The reciprocity symmetry of the S-matrix implies that

$$R_{ii}(\Phi) = R_{ii}(-\Phi), \quad T_{ij}(\Phi) = T_{ji}(-\Phi). \quad (5)$$

We now use these probabilities to determine the currents in the leads. Let the difference between the highest potential and the lowest potential be so small that the energy dependence of the transmission and reflection probabilities in this range can be neglected. It is convenient to introduce a fifth chemical potential μ_0 which is smaller than or equal to the lowest of the four potentials μ_i . Below μ_0 the states with negative and positive velocity are filled, and zero net current flows in each of the leads. We only need to consider the energy range $\Delta\mu_i = \mu_i - \mu_0$ above μ_0 . The reservoir i injects a current $ev_i(dn_i/dE)\Delta\mu_i$ into the lead i . Here v_i is the velocity at the Fermi energy in lead i , and $dn_i/dE = 1/2\pi\hbar v_i$ is the density of states for carriers with negative or with positive velocity at the Fermi energy. Thus the current injected by reservoir i is $(e/h)\Delta\mu_i$. Consider the current in lead 1. A current $(e/h)(1 - R_{11})\Delta\mu_1$ is reflected back to reservoir 1. Carriers which are injected by reservoir 2 into lead 2 reduce the current in lead 1 by $-(e/h)T_{12}\Delta\mu_2$. Similarly, from the current fed into leads 3 and 4 we obtain in lead 1 a current $-(e/h)(T_{13}\Delta\mu_3 + T_{14}\Delta\mu_4)$. Collecting these results and applying similar considerations to determine the currents in the other leads yields

$$I_i = (e/h) \left[(1 - R_{ii})\mu_i - \sum_{j \neq i} T_{ij}\mu_j \right]. \quad (6)$$

Note that these currents are independent of the reference potential μ_0 , since the coefficients multiplying the potentials add to zero. If we write Equation (6) in matrix form, both the rows and the columns of this transmission/reflection matrix add to zero (current conservation).

Let us generalize these results and assume that the perfect leads have many states at the Fermi energy. In leads with a cross section we have to consider both the motion of carriers across the lead and the motion along the lead. Motion in the transverse direction is quantized and characterized by a set of discrete energies, E_n , $n = 1, 2, \dots$. To this energy we have to add the kinetic energy for motion along the direction of the lead, $\hbar^2 k^2/2m$, such that $E_F = \hbar^2 k_n^2/2m + E_n$. For each energy E_n , which is smaller than E_F , we obtain two states at the Fermi energy (quantum channel). Each lead is thus characterized by a number N_i of quantum channels. The scattering matrix now contains

$$\left(\sum_{i=1}^{i=4} N_i \right)^2$$

elements denoted by $s_{ij,mn}$. Such an element gives the transmission amplitude for a carrier incident in channel n in lead j to reach channel m in lead i . The reciprocity of the S-matrix is now stated as $s_{ij,mn}(\Phi) = s_{ji,nm}(-\Phi)$. The probability for a carrier incident in channel n in lead i to be reflected into the same lead into channel m is denoted by $R_{ii,mn} = |s_{ii,mn}|^2$, and the probability for a carrier incident in lead j in channel n to be transmitted into lead i into channel m is $T_{ij,mn} = |s_{ij,mn}|^2$. Following [21], we assume that the reservoir feeds all channels equally up to the chemical potential μ_j . Furthermore, as in [21], we assume that the current injected into a channel is incoherent with the current in other channels. The current injected into each channel is then $(e/h)\Delta\mu_j$ independent of the velocity and the density of states of this channel. The current in lead i due to carriers injected in lead j is

$$I_{ij} = -(e/h) \sum_{mn} T_{ij,mn} \Delta\mu_j.$$

Therefore, if we introduce the traces

$$R_{ii} = \sum_{mn} R_{ii,mn}, \quad T_{ij} = \sum_{mn} T_{ij,mn},$$

which have the symmetry properties given in Equation (5), we find for the currents flowing from the reservoirs toward the conductor,

$$I_i = (e/h) \left[(N_i - R_{ii})\mu_i - \sum_{j \neq i} T_{ij}\mu_j \right]. \quad (7)$$

Here N_i is the number of channels in lead i . Recently, alternative derivations of Equation (7) have also been obtained [43–45]. Equations (6) and (7) provide the starting point for our subsequent discussion, giving the currents as response to the chemical potential differences between the reservoirs. Since the coefficients in Equation (7) obey Equation (5), they have the symmetry typical for linear response problems [4, 5]. The transmission probabilities in Equation (7) multiplied by e^2/h are the conductances which would be measured if the currents and potentials were measured simultaneously at all the probes. That, however, is not what is typically done in the experiments.

To derive the experimentally measured quantities, we proceed as in [7]. First let us connect Equation (7) to the (global) Onsager–Casimir symmetry relations. Casimir [4] considers a four-probe conductor (see Figure 1) where a current I_1 is fed into lead 1 and is taken out in lead 3, and a current I_2 is fed into lead 2 and leaves the sample through lead 4. Thus, we have to solve Equation (7) with the condition that $I_1 = -I_3$ and $I_2 = -I_4$. The result of such a calculation expresses the two currents as a function of differences of voltages $V_i = \mu_i/e$,

$$I_1 = \alpha_{11}(V_1 - V_3) - \alpha_{12}(V_2 - V_4), \quad (8)$$

$$I_2 = -\alpha_{21}(V_1 - V_3) + \alpha_{22}(V_2 - V_4). \quad (9)$$

The conductance matrix α_{ij} is expressed in terms of transmission probabilities in Appendix A. The diagonal elements are symmetric in the flux $\alpha_{11}(\Phi) = \alpha_{11}(-\Phi)$, $\alpha_{22}(\Phi) = \alpha_{22}(-\Phi)$, and the off-diagonal elements satisfy $\alpha_{12}(\Phi) = \alpha_{21}(-\Phi)$.

There are two additional ways of feeding two currents into the conductor of Figure 1. For each of these possibilities the currents and chemical potentials are as in Equations (8) and (9), related by a conductance matrix β_{ij} and γ_{ij} . The β and γ conductances have the same symmetry as the α conductances, but are generally not equal to these. The α , β , and γ conductances obey the *global* Onsager-Casimir symmetry relations referred to in the Introduction. These global symmetries for the conductances are more fundamental than the symmetry of the local conductivity tensor, Equation (2). The global symmetries hold even in situations where Equation (2) is not valid.

Now we can derive the resistance from Equations (8) and (9). In a four-probe setup only two of the chemical potentials are measured. Suppose the current flows from lead 1 to lead 3. The potentials measured are $\mu_2 = eV_2$ and $\mu_4 = eV_4$ under the condition that the current in leads 2 and 4 is zero. Taking $I_2 = 0$ in Equation (9) yields $V_2 - V_4 = (\alpha_{21}/\alpha_{22})(V_1 - V_3)$, and by using this in Equation (8) the current I_1 can be expressed as a function of $V_1 - V_3$. Thus, in this configuration the measured resistance is

$$R_{13,24} = (V_2 - V_4)/I_1 = \frac{\alpha_{21}}{(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}. \quad (10)$$

Since α_{21} is in general not symmetric, the resistance $R_{13,24}$ is also not symmetric. This result, however, is completely compatible with the (global) Onsager-Casimir symmetry relations. The point is that we are measuring an off-diagonal Onsager coefficient and not a diagonal element. It is V_2 and V_4 which determine the voltage drop across the sample, and not V_1 and V_3 . Now we switch the current and the voltage leads but keep the flux fixed. This means that I_1 in Equation (7) is zero. This yields a resistance

$$R_{24,13} = \alpha_{12}/(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}). \quad (11)$$

The sum of these resistances, $S_\alpha = (R_{13,24} + R_{24,13})/2$, is symmetric, due to the Onsager-Casimir relation $\alpha_{21}(\Phi) = \alpha_{12}(-\Phi)$.

For a given flux we find in general six resistances,

$$R_{mn,kl} = (h/e^2)(T_{km}T_{ln} - T_{kn}T_{lm})/D, \quad (12)$$

which differ in magnitude. $D = (h/e^2)^2(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})/S$ is a subdeterminant of the matrix defined by Equation (7). S is defined in Appendix A. All subdeterminants D of this matrix are equal and symmetric in the flux due to current observation. D is independent of the indices mn, kl . The resistances given by Equation (12) obey $R_{mn,kl} = -R_{mn,lk} = -R_{nm,kl}$, and, more fundamentally, the reciprocity relation (3). The six resistances, which differ in magnitude, can be

grouped into three pairs, each pair associated with one of the possibilities for feeding two currents into the four-probe conductor as discussed above. Thus, in addition to S_α we also have the combinations $S_\beta = (R_{14,32} + R_{32,14})/2$ and $S_\gamma = (R_{12,43} + R_{43,12})/2$, which are symmetric in the flux. Taking into account the symmetries of the transmission probabilities, Equation (5), we can now extend a relation, known in the classical case in the absence of a field [2], to our phase-coherence problem in the presence of a flux, and show that

$$S_\alpha + S_\beta + S_\gamma = 0. \quad (13)$$

The key result of this paper is Equation (12). An interesting property of Equation (12) is that the resistance measured in a four-probe setup is not necessarily positive. Resistances which change sign as the sample is rotated in the external magnetic field have indeed been measured by Soethout et al. [36], and were understood to be a property of a four-terminal measurement. Resistances which change sign as the magnetic field is increased have been observed in submicron structures by Timp et al. [52], but are interpreted as a "dynamic" phenomenon. Negative resistances are possible in a four-terminal resistance measurement. The resistance measured is not the total resistance of the sample, which is, of course, positive. Indeed, the total joule heat W produced by the conductor is

$$W = \frac{1}{e} \sum_i I_i \Delta\mu_i = \frac{1}{2h} \sum_{i < j} (T_{ij} + T_{ji})(\mu_i - \mu_j)^2. \quad (14)$$

To derive Equation (13) we have used Equation (7). Thus W is positive and is determined by the part of the transmission coefficients T_{ij} which is symmetric with regard to flux reversal.

3. Special limits of Equations (7) and (12)

In this section we discuss the application of the approach outlined above to a number of special cases.

• Two-terminal conductance

Current conservation in a two-port conductor requires $N_1 = R_{11} + T_{12}$ and $N_1 = R_{22} + T_{21}$, where N_1 and N_2 are the numbers of channels to the left and right. Using these relations, we see that the symmetry of the reflection coefficients implies $T_{12}(\Phi) = T_{12}(-\Phi)$ and $T_{21}(\Phi) = T_{21}(-\Phi)$. From this and Equation (5), we find that $T \equiv T_{12} = T_{21}$. The transmission coefficient of a two-port conductor is thus symmetric,

$$T(\Phi) = T(-\Phi). \quad (15)$$

Equation (6) or Equation (7) with $i = 1, 2$ and $T \equiv T_{12} = T_{21}$ yields a two-terminal conductance,

$$G = eI/(\mu_1 - \mu_2) = (e^2/h)T. \quad (16)$$

Therefore, Equation (15) implies that the two-terminal

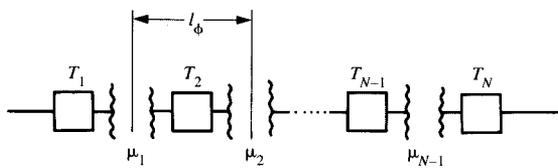


Figure 2

Model of a long wire: The phase is randomized in the reservoirs which are an inelastic length l_ϕ apart. From [38], reprinted with permission.

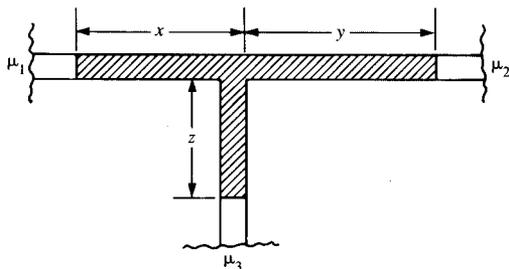


Figure 3

Three-probe conductor. The extra lead (z segment) connected to a reservoir can be viewed as a model of an inelastic scatterer or, alternatively, as a voltage probe. From [40], reprinted with permission.

conductance is symmetric with regard to flux or magnetic field reversal. All experiments known to us which are genuine two-probe experiments obey this symmetry.

Equation (16) was obtained in [30] in the limit of a large number of channels. The connection of Equation (16) with the Greenwood-Kubo formulation has been explored by Fisher and Lee [33]. This two-terminal formula has recently been used to discuss conductance fluctuations [48, 53–55] (variations of the conductance from sample to sample due to the microscopic disorder configuration in macroscopically identical conductors). On the basis of Equation (16), these fluctuations have been found to be universal in the (metallic) diffusive limit, i.e., to be independent of the degree of disorder and only weakly dependent on the geometrical shape of the conductor. The “universality” critically hinges on the simple relation between transmission and

conductance given by Equation (16), and breaks down if more sophisticated expressions for the conductance (or resistance) such as Equation (12) are invoked [40–46, 49–50].

To describe the conductance of a piece of wire much longer than the phase-breaking length l_ϕ , with the help of Equation (16), we imagine the wire divided into segments of length l_ϕ . Inelastic scattering cannot be neglected and is concentrated into reservoirs spaced a distance l_ϕ apart, as shown in Figure 2. This procedure, of course, is an approximation to reality, since inelastic scattering occurs uniformly in the bulk. Reference [38] uses this approach to show that the resistance of a one-dimensional conductor as a function of increasing inelastic scattering passes through a minimum. Furthermore, such a simple model allows an easy prediction of the size of conductance fluctuations or voltage fluctuations for voltage probes which are separated by a distance that is large compared to the phase-breaking length [12, 15]. Figure 2 represents a physical picture only if the portions of the conductor adjacent to the segment under study do act like reservoirs. This is not obvious. In the adjacent segments, in narrow wires, the current is, of course, not zero as it is in a reservoir. For small currents, as long as the currents and voltages are related linearly, that seems unimportant. Other features, such as the amplitude of the Aharonov-Bohm oscillations, depend on the fact that inelastic scattering occurs continuously throughout the conductor [56]. Therefore, there is a need for a conductance formula which takes into account the fact that most carriers traversing a segment of length l_ϕ have suffered at least one phase-randomizing event [38].

• The three-port conductor

A number of important insights can be gained by considering a conductor with three probes [1, 38]. It is considerably simpler to discuss such a conductor than the four-probe conductor described in [7] and Section 2. Consider the conductor shown in Figure 3, and let us focus on the situation where probe 3 is used to measure the chemical potential μ_3 . From Equation (6) or Equation (7) with $i = 1, 2, 3$, we find the chemical potential in lead 3 by taking $I_3 = 0$,

$$\mu_3 = \frac{T_{31}\mu_1 + T_{32}\mu_2}{T_{31} + T_{32}}. \quad (17)$$

Note that for $\mu_1 > \mu_2$ the chemical potential μ_3 is always between the two chemical potentials which drive the current through the conductor, $\mu_1 \geq \mu_3 \geq \mu_2$. Furthermore, $\mu_3(\Phi)$ is neither symmetric nor antisymmetric with regard to flux reversal. We can now use probe 3 to measure the potential differences $\mu_1 - \mu_3$ and $\mu_3 - \mu_2$ and can calculate the resistances

$$\mathcal{R}_{12,13} = (\mu_1 - \mu_3)/eI = \left(\frac{h}{e^2}\right) \frac{T_{32}}{D}, \quad (18)$$

$$R_{12,32} = (\mu_3 - \mu_2)/eI = \left(\frac{h}{e^2}\right) \frac{T_{31}}{D}, \quad (19)$$

with

$$D = T_{21}T_{31} + T_{21}T_{32} + T_{31}T_{23}. \quad (20)$$

D is a subdeterminant of the matrix defined by Equation (6) or Equation (7), and is invariant under flux reversal. Thus the two resistances given by Equations (18) and (19) are determined by the symmetry of T_{32} and T_{31} ; i.e., these resistances are neither symmetric nor asymmetric under flux reversal. However, the combined resistance (the two-terminal resistance)

$$R_{12,12} = (\mu_1 - \mu_2)/eI = \left(\frac{h}{e^2}\right) \frac{T_{31} + T_{32}}{D} \quad (21)$$

is symmetric. Thus, with regard to the two-terminal conductance, the fact that we have an additional lead does not change the symmetry. That important feature of our formulation of resistances is taken up again in Section 4.

The two-terminal conductance in the presence of an additional lead $\mathcal{G} = (R_{12,12})^{-1}$ differs now from Equation (16); it is given by

$$\mathcal{G} = \left(\frac{e^2}{h}\right)(T_{el} + T_{in}), \quad (22)$$

where the elastic transmission probability describing the transmission of carriers which emanate from port 1 and end up in port 2 without ever entering reservoir 3 is given by

$$T_{el} = T_{21}. \quad (23)$$

The inelastic transmission probability T_{in} describes carriers which emanate from port 1, reach reservoir 3 (where their energy and phase are randomized), and from reservoir 3, in an additional step, reach reservoir 2. Comparing Equation (22) with Equation (21) yields

$$T_{in} = \frac{T_{31}T_{23}}{T_{31} + T_{32}}. \quad (24)$$

Thus, the additional lead connected to an electron reservoir acts like an inelastic scatterer. Equation (22) allows us to describe the continuous transition from completely coherent transmission through the conductor to completely incoherent or sequential transmission. In the limiting case of completely coherent transmission, carriers are not allowed to enter reservoir 3. Consequently, $T_{13} = T_{32} = 0$, and the two-terminal conductance is given by Equation (16); i.e., $T_{el} = T$ and $T_{in} = 0$. In the limit of completely incoherent transmission, $T_{21} = T_{el} = 0$. In this case Equation (21) yields

$$R_{12,12} = \mathcal{G}_{12,12}^{-1} = \frac{h}{e^2} \left(\frac{1}{T_{31}} + \frac{1}{T_{23}} \right). \quad (25)$$

To obtain the latter result we have made use of the fact that for $T_{12} = 0$ we have $T_{23}(\Phi) = T_{23}(-\Phi)$. Equation (25) is

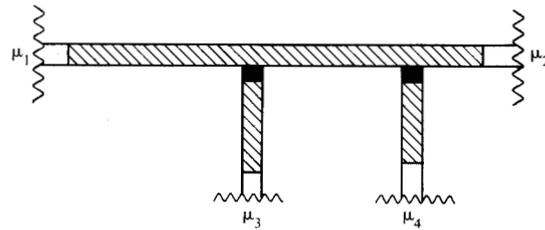


Figure 4

Four-terminal conductor with tunneling barrier junctions (dark areas) for two of the probes.

nothing but the classical addition of series resistors.

Equations (15)–(25) extend results presented in [38–40] for a sample without flux to the more general case of a three-probe conductor subject to a flux.

Adding a lead away from a conductor and connecting it to a reservoir gives us a simple way of introducing inelastic scattering into the conduction process. This method was used in [37] to study the effect of phase-randomizing events on the persistent current in a small normal loop and to investigate the dissipative response to a small oscillating flux superimposed on a static flux. Reference [38] investigates the transmission as a function of increasing inelastic scattering through a series of closely spaced barriers; and the effect of phase-randomizing events on resonant transmission through a double barrier is discussed in [39]. Thus, additional leads allow a double interpretation: They can be thought of as voltage probes, and they can be introduced as inelastic scatterers. Sample-specific fluctuations of voltage, resistance, and conductance in a three-probe conductor are the subject of [40, 45, 50].

• Point contacts

Further progress in lithography will make it possible to produce samples with leads made from different materials. An interesting possibility is the fabrication of barriers which separate the conductor and the leads.* This produces current leads [14, 20] and/or voltage leads which are weakly coupled [37–39] to the conductor.

Let us consider the conductor shown in Figure 4. Current is fed in at probe 1 and taken out at probe 2. Probes 3 and 4 serve to measure the voltage and are weakly coupled via tunneling barriers to the conductor. The probabilities for transmission from a probe into the conductor and into reservoir 1 or 2 are small because of the intervening

* S. Kaplan and C. Umbach, private communication.

tunneling barrier. Suppose the largest of these transmission probabilities is ϵ . The transmission probabilities of the conductor of Figure 4 can be expanded with respect to the small parameter ϵ . Transmission from port 1 to port 2 can proceed via paths which never cross the tunneling barrier, and thus $T_{12} = T_{12}^{(0)} + \dots$ to lowest order. The upper index in parentheses indicates the order in ϵ . But transmission from port 1 to port 3, for example, is only possible by crossing a barrier, and thus $T_{13} = \epsilon T_{13}^{(1)} + \dots$. Transmission from probe 3 to probe 4 requires that the barriers be traversed at least twice, and hence $T_{34} = \epsilon^2 T_{34}^{(2)} + \dots$. Via current conservation, the reflection coefficients can be expressed in terms of the transmission probabilities. Evaluation of Equation (12) yields a resistance

$$\mathcal{R}_{12,34} = \left(\frac{h}{e^2}\right) \frac{1}{T} \frac{T_{31}^{(1)}T_{42}^{(1)} - T_{32}^{(1)}T_{41}^{(1)}}{(T_{31}^{(1)} + T_{32}^{(1)})(T_{41}^{(1)} + T_{42}^{(1)})}. \quad (26)$$

$T \equiv T_{21}^{(0)} = T_{12}^{(0)}$ has the symmetry of the transmission probability of a two-terminal conductor given by Equation (15). The sums $T_{31} + T_{32}$ and $T_{41} + T_{42}$ are also symmetric with regard to flux reversal, since T_{34} is zero to order ϵ . Therefore, Equation (26) has precisely the symmetry required by the reciprocity theorem, Equation (3). Using Equation (12), it is now easy to calculate the resistances which are measured if current is fed and removed differently and the remaining "good and bad" contacts are used to measure the voltage. For the conductor of Figure 4, this yields six resistances which satisfy the sum rule, Equation (13). Equation (26) shows that the resistance, even when measured with point contacts, depends on the details of the coupling of the contacts to the conductor. *The resistance is not determined by the properties of the conductor alone (zero transmission probability through the contacts), but depends explicitly on how carriers can enter and leave the conductor through the probes.*

Instead of using Equation (7) we can derive Equation (26) in the following way: The voltage at probes 3 and 4 can be calculated by using Equation (17). To the lowest order in ϵ , the existence of one probe does not affect what is measured at the other. We thus find

$$\mu_3 = \frac{T_{31}\mu_1 + T_{32}\mu_2}{T_{31} + T_{32}} \quad (27)$$

and

$$\mu_4 = \frac{T_{41}\mu_1 + T_{42}\mu_2}{T_{41} + T_{42}}. \quad (28)$$

Here we have omitted, for simplicity of notation, the upper indices on the transmission probabilities indicating the order in ϵ . The measured potential difference is

$$\mu_3 - \mu_4 = \frac{T_{31}T_{42} - T_{32}T_{41}}{(T_{31} + T_{32})(T_{41} + T_{42})} (\mu_1 - \mu_2). \quad (29)$$

The net current through the conductor from probe 1 to

probe 2 is to lowest order in ϵ unaffected by these probes. Thus the current is $I = (e/h)T(\mu_1 - \mu_2)$. Using this and Equation (29) yields the resistance given by Equation (26). Let us again emphasize the possibility of measuring negative resistances in a four-terminal setup. Both μ_3 and μ_4 are bounded by μ_1 and μ_2 ; i.e., $\mu_1 \geq \mu_3 \geq \mu_2$ and $\mu_1 \geq \mu_4 \geq \mu_2$, but μ_3 is not necessarily greater than μ_4 . Thus, the only general bound we can give for the measured potential difference is $|\mu_3 - \mu_4| \leq \mu_1 - \mu_2$. Hence, the measured resistance $\mathcal{R}_{12,34}$ for the conductor of Figure 4 has upper and lower bounds given by the two-terminal resistance,

$$-\left(\frac{h}{e^2}\right) \frac{1}{T} \leq \mathcal{R}_{12,34} \leq \left(\frac{h}{e^2}\right) \frac{1}{T}. \quad (30)$$

We return to the subject of negative four-terminal resistances in Section 5.

• The Landauer formula

A very often quoted formula for the resistance of a one-dimensional conductor is the Landauer formula [16, 28, 29, 57]

$$\mathcal{R} = (h/e^2)(R/T). \quad (31)$$

How does this result relate to the resistance formulae discussed above? A four-terminal interpretation of Equation (31) has been put forth by Engquist and Anderson [32]. To arrive at Equation (31) they not only assume that the voltage probes are weakly coupled, as discussed above, but also assume that the probes couple to the conductor in a symmetric fashion with regard to right- and left-moving carriers. Furthermore, at the junction of the conductor with the probe, they match the currents and not the current amplitudes. In contrast, our treatment is fully quantum-mechanical. We also note that the situation envisioned by these authors differs from that of Figure 4 in that the voltage probes are connected to the perfect leads and the conductor is disordered only *between* the voltage probes. Elastic scattering is then characterized by a transmission and a reflection probability T and R of the disordered region. These simplifications and assumptions give rise to transmission probabilities $T_{12} = T_{21} = T$ to order ϵ^0 , $T_{31} = T_{13} = T_{42} = T_{24} = 1 + R$, $T_{32} = T_{23} = T_{14} = T_{41} = T$ to order ϵ , and $T_{43} = T_{34} = T$ to order ϵ^2 . Using this in Equation (29) yields

$$\begin{aligned} \mu_3 - \mu_4 &= \frac{1}{4}[(1+R)^2 - (1-R)^2](\mu_1 - \mu_2) \\ &= R(\mu_1 - \mu_2). \end{aligned} \quad (32)$$

Since the current is $I = (e/h)T(\mu_1 - \mu_2)$, this gives Equation (31). Thus, by using the assumptions of Engquist and Anderson, we can deduce the Landauer formula from Equation (12) or Equation (26). T has the symmetry of the transmission probability of a two-terminal conductor. Since $R = 1 - T$, the Landauer result is symmetric under flux

reversal. Instead of Equation (3), the Engquist and Anderson discussion yields $R_{mn,kl}(H) = R_{kl,mn}(H) = R_{mn,kl}(-H)$; i.e., it is not necessary to exchange current and voltage leads when reversing the magnetic field. For a four-probe conductor, we typically have six differing resistances, whereas the Engquist and Anderson discussion yields only three. In addition to $R_{12,34}$ given by Equation (31), using Equation (12) and the transmission probabilities as specified above, we find $R_{14,23} = -(h/e^2)[R/T]$ and $R_{13,42} = 0$. The sum of these three resistances, according to Equation (13), is zero. (Note that in the latter configuration, the voltage difference is not measured along a piece of the conductor carrying a net current. Such voltage differences across Y-shaped leads are generally not zero and have indeed been measured [41]. For another experiment demonstrating such nonlocal effects, see [58].) In principle it could have been possible that for strictly one-dimensional conductors a higher symmetry than that predicted by Equation (3) applies. But Equations (12) and (26), which are valid independent of the number of channels, tell us otherwise. Note also that the symmetry assumed by Engquist and Anderson on the coupling of the voltage probe with the conductor ensures that the resistance, Equation (31), is always positive, in contrast to Equation (26). We return to the discussion of the Landauer formula from a different point of view in Section 5. The symmetry of the conductance formulae of Azbel [25] and Büttiker et al. [21] is discussed in Appendix B.

4. Rigidity of the reciprocity symmetry

The situation discussed until now is highly conceptual. We have discussed conductors with four leads which are connected to reservoirs so close to the conductor that it can be assumed that scattering within the conductor is only elastic. Clearly, the spatial separation of elastic and inelastic scattering is more a theorist's invention than an experimental reality. In the experiments, the probes lead away from the section of conductor which is under study. The probes are connected to macroscopic pads over distances which are large compared to the inelastic scattering length. Thus, inelastic scattering occurs in a rather uniform fashion throughout the conductor and the leads. Using the concepts developed here, we would like to understand why reciprocity is also observed in the presence of inelastic scattering. Furthermore, reciprocity is also observed regardless of how many probes are connected to the conductor. We have already pointed out that probes act like inelastic scatterers. Thus the validity of the reciprocity relations both in the presence of inelastic scattering and in the presence of an arbitrary number of leads attached to the conductor is really the very same problem.

Below we show that the relations of Equations (6) and (7) can be generalized and are valid for transmission probabilities which are the sum of an elastic coherent part (el) and an inelastic or incoherent part (in). Thus, in general,

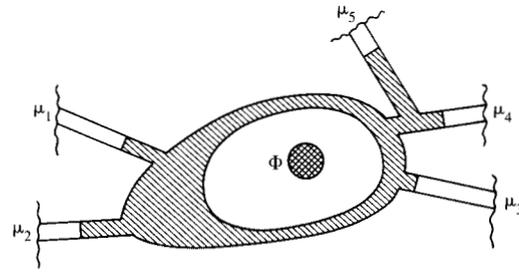


Figure 5

Five-probe conductor. The addition of a lead does not change the reciprocity symmetry and permits us to study conductors which are large compared to the phase-breaking length.

$$\hat{T}_{ij}(\Phi) = T_{el,ij}(\Phi) + T_{in,ij}(\Phi), \quad (33)$$

where the combined transmission probability \hat{T}_{ij} has exactly the symmetry given by Equation (5). Similarly, the reflection coefficients are in general a sum of both an elastic and an inelastic part, and the symmetry of the total reflection probability is again given by Equation (5).

Consider now the conductor shown in Figure 5, where we have added a fifth probe connected to a potential μ_5 . The currents in this conductor are determined by Equation (7), where $i = 1, \dots, 5$. Here we want to show that by eliminating one of the chemical potentials, say μ_5 , we once again obtain Equation (7), with $i = 1, \dots, 4$ and T_{ij} replaced by \hat{T}_{ij} . This then shows that Equation (12) is still valid; the only difference is that the T are replaced by \hat{T} . If probe 5 is a voltage probe or an inelastic scatterer, we must require that $I_5 = 0$. This condition determines μ_5 as a function of the remaining chemical potentials,

$$\mu_5 = \frac{1}{(N_5 - R_{55})} \sum_{j=1}^{j=4} T_{5j} \mu_j. \quad (34)$$

Using Equation (34) to eliminate μ_5 in the equations for the currents at the other probes yields

$$I_i = \frac{e}{h} \left[(N_i - \hat{R}_{ii}) - \sum_{j=1}^{j=4} \hat{T}_{ij} \mu_j \right], \quad (35)$$

with

$$\hat{R}_{ii} = R_{ii} + \frac{T_{i5} T_{5i}}{N_5 - R_{55}}, \quad (36)$$

$$\hat{T}_{ij} = T_{ij} + \frac{T_{i5} T_{5j}}{N_5 - R_{55}}. \quad (37)$$

In Equations (36) and (37), the first term gives the reflection

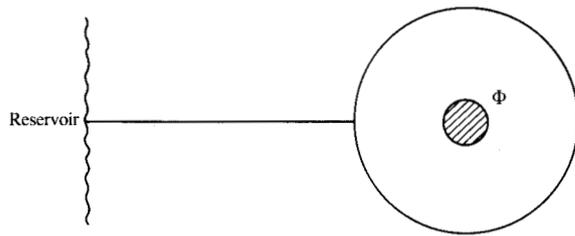


Figure 6

Normal loop penetrated by an Aharonov-Bohm flux supports a circulating equilibrium current despite the coupling to a phase-breaking dissipative reservoir. From [37], reprinted with permission.

(transmission) of carriers which have *not* entered reservoir 5, and the second term gives the reflection (transmission) of carriers which have entered reservoir 5. Since the reservoirs act as phase-breakers, the second term in Equations (36) and (37) can be viewed as the inelastic, incoherent part of the total reflection (transmission) probability. The first term in Equations (36) and (37), on the other hand, describes coherent reflection (transmission). Thus Equations (36) and (37) appear as simple generalizations of the relations found for a three-probe conductor [38–40].

Now it is easy to see that the symmetry of the new transmission and reflection probabilities \hat{T} and \hat{R} is the same as that of the original purely elastic transmission probabilities. Furthermore, by using Equation (7) for a five-probe conductor and eliminating μ_5 , we obtain

$$N_i = \hat{R}_{ii} + \sum_{j=1}^4 \hat{T}_{ij} \quad (38)$$

and

$$N_i = \hat{R}_{ii} + \sum_{j=1}^4 \hat{T}_{ji}. \quad (39)$$

Equations (38) and (39) are a consequence of current conservation in a four-probe conductor. Therefore, the \hat{T} and \hat{R} satisfy the same symmetry conditions and current-conservation relations as the T and R . Hence, all results obtained for the T and R in Sections 2 and 3 of this paper are also valid for the \hat{T} and \hat{R} .

Obviously, if we have a conductor with many leads $n > 4$, we can repeat the steps outlined above $n - 4$ times, until only the chemical potentials of the particular four-probe measurement under study occur. Equation (12) is then valid for a set of generalized transmission and reflection probabilities. Thus, these considerations show that the

reciprocity symmetry is a very *rigid* feature of electrical conduction. Reciprocity applies regardless of the number of leads attached to the conductor and regardless of whether we deal with elastic or inelastic transmission.

5. Self-induced fields

Charge transport gives rise to magnetic and electric fields. In the presence of steady current flow, considered in this paper, current density and charge density are related to fields via

$$c \nabla \times B = j, \quad (40)$$

$$\nabla D = 4\pi e(\rho - \rho_0). \quad (41)$$

In this section we are chiefly concerned with the induced electric field, or with the associated electrostatic potential U :

$$\nabla^2 U + (4\pi e/e_L)(\rho - \rho_0) = 0. \quad (42)$$

The key intention is to explore the connection of the electrostatic potential $U(r)$, which is defined at every point in the conductor with the chemical potentials measured at the contacts, as discussed in the previous sections. Before discussing this, it is worthwhile to consider briefly the induced magnetic fields.

• Magnetic field induced by persistent currents

Consider the conductor in Figure 1 at equilibrium. All the reservoirs are at the same chemical potential, $\mu = \mu_1 = \mu_2 = \mu_3 = \mu_4$. According to Equation (7), this implies that the currents at all the contacts are zero, $I_1 = I_2 = I_3 = I_4 = 0$. However, this does not imply that the current density $j(r)$ is zero in the interior of the conductor. Simple model calculations suggest that there exists an equilibrium current pattern $j(r)$ in tiny and open conductors. Figure 6 shows an open conductor. A loop is connected to a reservoir via a single lead. The lead allows an exchange of carriers between the loop and the reservoir. As shown in [37], such a loop exhibits a persistent current which is a periodic function of the flux Φ threading the loop. The amplitude of the circular equilibrium current is nonzero as long as carriers can complete a full revolution before escaping from the loop into the reservoir. This simple model was introduced to discuss the effect of inelastic events on the circular currents found in closed loops in [18]. The simple model of Figure 6 demonstrates that circular equilibrium currents are not a property of closed systems only. Considering the lead connected to a reservoir as an inelastic scatterer, [37] arrives at the same conclusion as [59]: A modest number of inelastic events only reduces the amplitude of the circular currents but does not suppress such currents completely. Comparison [8, 10] of the inelastic effects introduced via a current lead coupled to a bath [37] with intrinsic inelastic effects in a closed loop [59] highlights the effect of coupling a measurement probe to a tiny conductor. For additional work on closed loops we refer to [60]. In the conductor of Figure 6, an equilibrium current pattern arises because elastic

scattering leads to a density of states which is sharply peaked near the eigenstates of the closed loop. In a more general geometry, such as the conductor in Figure 1, carriers are not confined to move on a circular path. However, it is known that quantum-mechanical interference leads to an enhanced probability for a carrier which initially is at point P to return to point P (see for example [15]). A net current arises if the probabilities for clockwise and anticlockwise motion along such a path are different. Since the net current through the contacts is zero, no joule heat is produced. Calculation of these currents requires knowledge of the wave functions at every point in the conductor. Moreover, since the currents are an equilibrium feature, they are not determined by the states at the Fermi energy alone but require knowledge of the wave functions at all energies. Typically, however, the main contribution to these currents arises from a narrow energy interval extending from the Fermi energy to an energy somewhat smaller. On a length scale, small compared to the phase-breaking length, we can thus expect to find a nonzero ensemble average $\langle [j(r)]^2 \rangle$. These currents induce a magnetic field according to Equation (40).

If the chemical potentials of the conductor of Figure 1 are different and a net current is induced, say from contact 1 to contact 2, then, as is well understood, an additional magnetic field $B(r; \mu_1 - \mu_2)$ is produced which is proportional to the difference of the chemical potentials. We do not discuss this further, but instead refer the reader to [42] and [61], which calculate such fields in the metallic diffusive regime.

• Induced electric fields

Induced electric fields in the presence of current flow past isolated impurities have been emphasized by Landauer [27, 28, 58, 62]. Local fields are of importance for the discussion of nonlinear effects [63, 64], and they play a central role in the von Klitzing effect [65]. Here we wish to stress the distinction between the local field $E(r)$, or the potential $U(r)$, and the chemical potentials μ_i discussed in the previous sections of this paper. The chemical potentials μ_i are thermodynamic potentials which characterize a bath. The potential $U(r)$, on the other hand, characterizes the distribution of unscreened charges in the conductor. In a macroscopic conductor, where each volume element also contains a large number of carriers, $U(r)$ also becomes a thermodynamic quantity, i.e., a local Fermi energy. In a macroscopic conductor, we can couple the volume element under consideration to a bath, and if the coupling is sufficiently "weak," the bath has a chemical potential $\mu(r) = eU(r)$, at least if $U(r)$ varies slowly compared to the screening length. In the coherent quantum transport regime we must be more cautious. It makes a difference whether a small conductor over a given length interval is closed, or has a junction to an extra lead, with a reservoir attached. In general, as we show, $\mu(r) \neq eU(r)$.

One-dimensional two-probe conductor

Consider a one-dimensional conductor consisting of a disordered region with perfect leads attached to each side. The perfect leads are in turn connected to reservoirs at chemical potentials $\mu_1 > \mu_2$. Now suppose that self-consistent screening applies for the equilibrium situation, i.e., for $\mu_1 = \mu_2$. Consider the additional charge density $\Delta\rho$ induced by the current flow. The density of states in the perfect leads, corresponding to states with positive velocity, is $dn/dE = 1/2\pi\hbar v$. We denote the wave function which describes carriers incident from reservoir 1 with chemical potential μ_1 by $\psi_1(x)$, and the wave function which describes carriers incident from the right by $\psi_2(x)$. The added charge is [37]

$$\Delta n = \frac{dn}{dE} (\mu_1 - \mu_2) |\psi_1(x)|^2, \quad (43)$$

where ψ_i is normalized such that the incident beam has unit amplitude. Both the wave function and the density of states are taken at the Fermi energy. In a region where Δn is spatially independent, this excess density is screened. Since the total charge density in such a region is zero, the screening field raises or lowers the band bottom to achieve this [62]. The charge which is gained or lost by adjusting the band bottom is

$$\Delta n_{\text{eq}} = \frac{dn}{dE} (eU - \mu_2) (|\psi_1(x)|^2 + |\psi_2(x)|^2). \quad (44)$$

Note that Δn_{eq} is the local excess density which builds up if the chemical potential of both reservoirs is raised by $eU - \mu_2$ from $\mu_1 = \mu_2$. The net charge difference $\Delta\rho$, which remains unscreened and enters Equation (41), is thus $\Delta\rho = \Delta n - \Delta n_{\text{eq}}$. Inserting this into Equation (41) yields a differential equation for $U(x)$:

$$\lambda^2 \nabla^2 eU(x) + [\mu_1 |\psi_1|^2 + \mu_2 |\psi_2|^2] - eU(x) (|\psi_1|^2 + |\psi_2|^2) = 0, \quad (45)$$

where $\lambda = (\epsilon_L/4\pi e^2)^{1/2} (dE/dn)^{1/2}$ is a screening length. The proper solution eU of Equation (45) has the value μ_1 in the left reservoir and drops to the chemical potential μ_2 in the reservoir to the right. Below, we resort, for simplicity, to drastic approximations. Suppose that the wave functions vary slowly on the scale λ , the screening length. In this case the first term in Equation (45), $\lambda^2 \nabla^2 U$, can be neglected, and we obtain

$$eU(x) = \frac{|\psi_1(x)|^2 \mu_1 + |\psi_2(x)|^2 \mu_2}{|\psi_1(x)|^2 + |\psi_2(x)|^2}. \quad (46)$$

If the screening length is not short compared to the Fermi wavelength, we can still salvage Equation (46). If we are only interested in the long-range variation of $U(x)$, we can average the wave functions in Equation (45) over distances of the order of λ . Clearly, if such an averaging procedure is used, some information on the phase sensitivity of $U(x)$ is lost. Equation (46) was obtained by Entin-Wohlman et al.

[35], who argue that $eU(x)$ is what is actually measured at a chemical potential probe.

Applied to the perfect leads, Equation (46) reproduces, with an additional approximation, the Landauer result, Equation (31). To see this, consider a scatterer connected to perfect leads. The scatterer has a transmission probability $T = |t|^2$ and a reflection probability $R = |r|^2$. To the left we have $|\psi_1|^2 = 1 + R$, $|\psi_2|^2 = T$, and from Equation (46) we find

$$eU_l = \frac{1}{2}[(1 + R)\mu_1 + T\mu_2]. \quad (47)$$

To the right we have $|\psi_1|^2 = T$, $|\psi_2|^2 = 1 + R$, and from Equation (46) we find

$$eU_r = \frac{1}{2}[T\mu_1 + (1 + R)\mu_2]. \quad (48)$$

Since the total current driven through the conductor is $I = (e/h)T(\mu_1 - \mu_2)$, we immediately find $\mathcal{R} = (U_l - U_r)/I = (h/e^2)(R/T)$.

In the derivation of the Landauer formula given above, we have related $|\psi|^2$ to the transmission and reflection probabilities. Since the reflected wave interferes with itself, the exact density of carriers is determined by $|\psi(x)|^2 = 1 + R + 2\text{Re}(re^{-ikx})$. Since the reflection amplitude r is proportional to $R^{1/2} \geq R$, it is actually the interference term which is dominant for weak elastic scattering ($R \ll 1$). If the exact expression for the charge densities is inserted into Equation (45), the result is a voltage which is oscillating even in a perfect lead. The suppression of such interference terms is one reason that Landauer's result is positive, whereas Equation (26), which allows for such interference terms, can give a negative resistance.

The voltage $U(x)$, as defined by Equation (46), does not match the chemical potentials of the reservoirs. (This can only be achieved by allowing the one-dimensional leads to spread out to accommodate a large density of states [16, 58, 66].) Imry [11] has pointed out that the difference in potential between a reservoir and a lead gives rise to a contact resistance, $\mathcal{R}_{\text{con}} = (h/e^2)(V_1 - U_1)/I$, where $\mu_1 = eV_1$. Using Equation (47), the contact resistance is, according to this interpretation, universally equal to $h/2e^2$ for a one-channel conductor. However, due to the oscillatory nature of the voltage in the lead, we can expect such contact resistances to fluctuate from sample to sample and to exhibit a sensitivity to the phase of the wave function. If we consider the conductor of Figure 4 and use the results of Section 3, we find contact resistances which can fluctuate in a wide range. If we define the contact potential as $\mu_1 - \mu_3$ and use Equation (27), we find a contact resistance with a lower bound of zero and an upper bound of $(h/e^2)T^{-1}$.

Equation (46) can be used to determine the voltage U at two arbitrary points along the conductor [35]. Denote these points by x_1 and x_2 . Then, by evaluating $\mathcal{R} = (U_1 - U_2)/I$ with the help of Equation (46), we find

$$\mathcal{R}(x_1, x_2) = \frac{h}{e^2} \frac{1}{T} \frac{|\psi_2(x_2)|^2 |\psi_1(x_1)|^2 - |\psi_2(x_1)|^2 |\psi_1(x_2)|^2}{[|\psi_1(x_1)|^2 + |\psi_2(x_1)|^2][|\psi_1(x_2)|^2 + |\psi_2(x_2)|^2]}. \quad (49)$$

Equation (49) is reminiscent of Equation (26). Like the Landauer formula, however, it does not contain any perturbations due to the leads. It links the electrical potential $U(x)$ to a "resistance." The "resistance" given by Equation (49) typically exhibits no symmetry at all with regard to flux reversal. If Equation (47) is applied to a one-dimensional ring [35, 67] and at least one of the points x_1 or x_2 lies in the disordered part of the conductor, the resistance given by Equation (49) can be expected to be neither antisymmetric nor symmetric with respect to flux reversal, despite the fact that the conductor is only connected to two reservoirs. This is demonstrated by a calculation in [67]. [Reference [67] makes an additional approximation and replaces the local equilibrium density of states with the density of states in the leads. This corresponds to setting the combined densities of the two wave functions multiplying U in Equation (45) equal to 2.] A similar calculation in [35] finds a purely antisymmetric resistance, due to the high symmetry of the conductor investigated and the symmetric location of the points x_1, x_2 chosen.

One-dimensional many-probe conductor

To better illustrate the difference between the voltage measured at a contact and U given by Equation (46), we consider for simplicity a *one-channel* conductor connected to three reservoirs, as shown in Figure 3. Let the density of states in all the leads be equal. Proceeding as above gives, for a three-probe conductor, a local voltage $U(r)$,

$$eU(r) = \frac{|\psi_1(r)|^2 \mu_1 + |\psi_2(r)|^2 \mu_2 + |\psi_3(r)|^2 \mu_3}{|\psi_1(r)|^2 + |\psi_2(r)|^2 + |\psi_3(r)|^2}. \quad (50)$$

Here the index on the wave function indicates the reservoir from which the carriers are injected into the conductor. Equation (50) is valid for every branch of the conductor with $r = x$, $r = y$, or $r = z$ on the corresponding branch (see Figure 3). Connecting a lead to the conductor thus changes the potential throughout the conductor. Equation (48) has been evaluated in the perfect leads of a three-probe conductor in [38]. Let us briefly consider probe 3, acting as a voltage probe, and show that Equation (17) results. In this case, within the *perfect* lead of probe 3, we have $|\psi_1(x)|^2 = T_{31}$, $|\psi_2(x)|^2 = T_{32}$, and $|\psi_3(x)|^2 = 1 + R_{33}$. Inserting this into Equation (48) and using Equation (7) for $i = 3$ with $I_3 = 0$ yields $eU_3 = \mu_3$. This is an astonishing result in view of the approximations made to arrive at Equation (50). For a voltage lead which does not support a net current, it might be more adequate to consider just the long-range variation of U (and thus to neglect the interference terms in the absolute

values of the wave functions). Clearly, these results can easily be extended to a four-probe conductor with one-dimensional perfect leads. In this case, the voltage $U(r)$ is determined by four wave functions and four chemical potentials. Thus, for the many-probe conductor with one-dimensional leads, we have an (approximate) picture of the connection of the local potential with the chemical potentials of the measuring baths. We see that the reciprocity symmetry is connected to the fact that at a probe (a point of measurement), three (or four) wave functions determine the outcome of the measurement. In contrast, only two wave functions are used in Equations (46) and (49).

Potential fluctuations in the measurement lead

Equation (50) leads to a picture of the local potential $U(r)$ along a three-probe conductor, as shown in Figure 7. A net current flows from the bath with chemical potential μ_1 to the bath with chemical potential μ_2 . The measurement probe, leading to a bath at a chemical potential μ_3 given by Equation (17), is connected to the conductor at point P . Equation (50) predicts a potential $U(r)$ which fluctuates along the probe depending on all three wave functions ψ_i . Fluctuations of $U(z)$ along the probe occur for several reasons. First, the probe itself is typically a disordered conductor. In this case the fluctuations are determined by the precise disorder configuration within a phase-breaking length. Even if the probe is an ideal perfect wire, a voltage variation occurs across the junction of the perfect probe with the conductor [38, 40]. Thus, in general, the chemical potential $\mu_3 = eV_3$ is not related in a simple way to the local potential $U_p \equiv U(P)$. The fact that U_p and V_3 are not equal is important. It means that a *contact potential difference* $V_C = U_p - V_3$ exists between the local potential at P and the measured voltage. Typically, voltage drops arise due to current flow past an obstacle. But in the conductor depicted in Figure 7, there is *no net* current flow in the measurement lead. A situation similar to that shown in Figure 7 also follows from the work of Maekawa et al. [43] and Kane et al. [45]. They define a local electric field by enforcing current conservation on an expression for the nonlocal current-field relation.

The existence of such contact potentials means that a voltage measurement with a lead does not give us direct information on the local potential of the conductor at the point of attachment. Such contact potentials arise not only in the metallic diffusive limit. Even if voltages were measured by inserting a tunnel junction between the conductor and the lead, or if the tunneling microscope were used to measure the potential [68], such contact potentials would also be present. In the presence of a large barrier, the wave functions ψ_i are exponentially attenuated at the junction. The wave functions ψ_1 and ψ_2 have an exponentially small amplitude factor in the measurement lead; ψ_3 , which is large in the probe, is exponentially small in the conductor. As shown by Equation (17), the measured

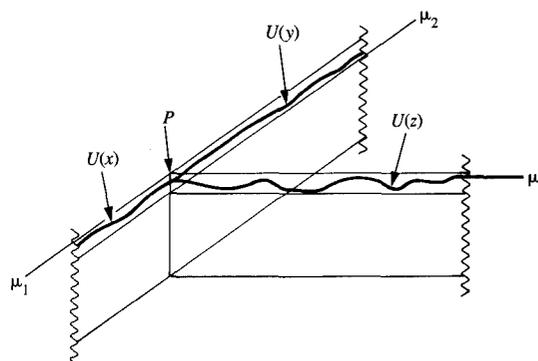


Figure 7

One-dimensional conductor with a voltage probe. Current flows from the bath at chemical potential μ_1 to the bath at chemical potential μ_2 . A voltage probe is attached to the conductor at point P . The measured chemical potential is μ_3 . The heavy line depicts the local potential U along the x , y , and z segments. Typically, the measured chemical potential differs from the electric potential U at the point P of attachment. Screening causes the band bottom (faint lines below μ_2) to follow $U(r)$.

voltage is determined by T_{31} and T_{32} , i.e., by the exponentially small wave functions ψ_1 and ψ_2 at the location of the bath. If the junction cannot be controlled on an atomic scale, the attenuation of ψ_1 and ψ_2 at the barrier varies exponentially from sample to sample. Hence the ratio T_{31}/T_{32} exhibits fluctuations that increase as the coupling between the lead and the conductor becomes weaker. Invoking tunnel junctions between the leads and the conductor can, therefore, be expected to increase the fluctuations in the contact potential and to lead to voltage fluctuations which are even bigger than those measured [41,42] in the metallic diffusive limit [40, 46, 49, 50]. Even metallic-diffusive voltage probes give rise to voltage fluctuations which increase with the length of the probe [40, 46].

Since the chemical potential of a measurement probe is not simply related to the voltage U at the point of attachment, measurement of negative resistance in the geometry of Figure 4 does not imply that the voltage U increases in the direction of the current flow. The contact potential V_C can be positive at probe 3 and negative at probe 4. If these contact potentials are large enough, we measure a negative resistance despite the fact that $U(r)$ drops monotonically along the conductor.

Voltages can be measured other than by the exchange of carriers with a probe; an alternative technique uses capacitive probes [57]. It is suggested that in the absence of particle exchange with the measurement probe we can

measure the local potential U_p . Thus, the appearance of contact potentials discussed previously is avoided. The key question which must be answered is this: Can we have enough Coulomb coupling between the conductor and a capacitor plate to make a measurement without significantly affecting the effective potential which the carriers see while moving along the conductor? A recent experiment which uses capacitive coupling to shift the phase of the Aharonov-Bohm oscillations [18-21] is described in [69].

Measurement-theoretical aspects

Equations (15) and (17) are also of interest for the theory of measurement [70]. We can look at these equations as describing the interaction between the measurement apparatus (the measurement probe with a bath attached at its end) and the system (the conductor). In contrast to the discussions of the measurement process, which treat a Hamiltonian for the combined system, the approach presented here describes the measurement as a scattering process and analyzes an S-matrix. The phase of the wave functions is broken by the reservoir at the end of the probe. This notion avoids a difficulty which measurement theory tries to explain: Why doesn't the measurement apparatus obey the superposition principle even though the total system is described by a Hamiltonian?

There is a fundamental difference in how we have treated the local potentials and the currents. The reciprocity symmetry is a consequence of microscopic reversibility. Current conservation and time-reversal invariance are what determine the symmetry of the transmission probabilities in Equations (6) and (7). In Equation (45) the time-reversed wave functions of ψ_1 and ψ_2 do not occur. The time-reversed wave function of ψ_1 describes carriers incident from both reservoirs which in the disordered region interfere, giving a beam of unit amplitude for the outgoing wave. The time-reversed wave functions are excluded, since carriers incident from different reservoirs are incoherent [18, 62].

Local potentials in a many-channel two-probe conductor

Consider a disordered region connected to two perfect leads. Assume, for simplicity, that the perfect leads are strips, with x the direction along the lead and y the direction perpendicular to the strip. Also assume that the perfect leads to the left and right are identical and support N quantum channels with a density of states $(dn_i/dE) = 1/2\pi\hbar v_i$. Here v_i is the velocity in channel i at the Fermi energy in the x direction. As in [21], we assume that the reservoir feeds channels incoherently with respect to one another. With these specifications the many-channel generalization of Equation (46) is

$$eU(x, y) = \frac{\sum_{i=1}^{i=N} \frac{1}{v_i} [|\psi_{1,i}(x, y)|^2 \mu_1 + |\psi_{2,i}(x, y)|^2 \mu_2]}{\sum_{i=1}^{i=N} \frac{1}{v_i} [|\psi_{1,i}(x, y)|^2 + |\psi_{2,i}(x, y)|^2]} \quad (51)$$

Here $\psi_{1,i}$ is the wave function at the Fermi energy describing carriers incident in channel i from the left-hand reservoir. Similarly, $\psi_{2,i}$ describes carriers incident in channel i from the right-hand reservoir. As in the one-channel case, the amplitude of these wave functions is normalized such that the incident wave has amplitude 1. The voltage $U(x, y)$ obtained in this manner is a complicated fluctuating function of x and y . Let us briefly discuss the voltage $U(x, y)$ in one of the perfect leads. In the perfect leads we can represent the wave functions ψ as a superposition of eigenstates of the lead Hamiltonian. This Hamiltonian is separable and has eigenstates $e^{\pm ik_j x} f_j(y)$. Here $f_j(y)$ is the "transverse eigenfunction" of channel j and k_j is the wave vector along the lead. The wave functions in the left perfect conductor are

$$\psi_{1,i} = \sum_{j=1}^{j=N} f_j(y) [\delta_{ij} e^{ik_j x} + (v_i/v_j)^{1/2} r_{11,ji} e^{-ik_j x}] \quad (52)$$

for carriers incident from the left, and

$$\psi_{2,i} = \sum_{j=1}^{j=N} f_j(y) (v_i/v_j)^{1/2} t_{12,ji} e^{-ik_j x} \quad (53)$$

for carriers incident from the right. Here $r_{11,ji}$ is the probability amplitude for reflection into channel j of a carrier incident in channel i . $t_{12,ji}$ is the transmission probability for carriers incident in the right perfect conductor in channel j to reach channel i to the left. The density of carriers $|\psi_{1,i}|^2$ contains diagonal terms $f_j(y)f_j^*(y)$ proportional to the density of the incident wave (in channel i) and the density of the reflected waves. There are off-diagonal terms $f_i(y)f_j^*(y)e^{i(k_i-k_j)x}$ proportional to the reflection amplitudes in channels i and j . Furthermore, there are $2N$ terms which arise from multiplying the incident wave with the reflected waves. These latter terms are proportional to $f_i(y)f_j^*(y)e^{-ik_j x}$. Hence the voltage U in the perfect leads is nonuniform and exhibits long-range oscillations, since $k_i - k_j$ can be small (of the order of k_F/N). Therefore, since screening typically occurs over much shorter distances, the voltage $U(x, y)$ follows these long-range oscillations.

References [21], [25], and [34] do not allude to spatially nonuniform voltages in the leads, but attribute a spatially averaged voltage to each lead. In these works the densities $|\psi|^2$ are averaged with respect to x and y . Thus the densities are expressed in terms of transmission and reflection probabilities alone. The potential drop across the disordered region is obtained by introducing these averaged densities into Equation (51). The result for the conductance is given in Appendix B [Equation (B7)]. We emphasize that using the spatially averaged densities in Equation (51) is not equivalent to calculating a spatial average of the exact voltage; i.e., $\langle eU \rangle = \left\langle \frac{eU}{\langle \psi^2 \rangle} \right\rangle \neq \frac{\langle eU \psi^2 \rangle}{\langle \psi^2 \rangle}$. Thus, the spatial average of Equation (53) is in general not equal to the potentials determined in [21, 25, 28, 29, 34]. To calculate the voltage drop across the disordered region, [21], [25], and

[34], following [28] and [29], treat the carrier densities classically.

To achieve a uniform voltage in a perfect lead, it is necessary to consider a lead that supports a number of channels which is large compared to $G/(e^2/h)$. In this case the amplitudes of the wave functions [Equation (53)] associated with the transmitted carriers are small. We can neglect the terms in Equation (51) proportional to μ_2 in the left perfect lead and can neglect the terms in Equation (51) proportional to μ_1 in the right perfect lead. Wide leads are needed if we want to construct a potential which smoothly joins the chemical potentials of the reservoirs.

6. Conclusions

The resistance formulae discussed in Sections 2–5 exhibit a certain beauty and elegance. Their agreement with the experimentally observed symmetries [27, 36] and their successful application [43–46, 49–50] makes one suspect that they will likely survive more realistic treatments of, for instance, the reservoirs. We have first derived expressions for quantum-mechanical coherent transmission. In physically relevant situations, we deal with coherent and incoherent transmission. Our expressions are also applicable in this case, and permit us to study the continuous transition from completely coherent to completely incoherent transmission [38, 39]. We have emphasized that resistances measured at contacts relate chemical potentials and currents.

Furthermore, our discussion stresses that the measured resistance depends on the properties of the contacts, whether we deal with good contacts or with point contacts.

The discussion of the local electric potentials given in Section 5 seems much more susceptible to the detailed assumptions which we have made. Realistically, a reservoir feeds carriers into the conductor not in a continuous coherent fashion but with finite coherence length and with fluctuations in time. That has little effect on the total (time-averaged) current, and it is only currents which determine the resistances of Sections 2–5, but it is likely to alter the charge accumulated in some small spatial region. The detailed distribution of charge and voltage is, however, of interest and provides a physically appealing picture. We hope, therefore, that the problems exposed in Section 5 will stimulate further research in this direction.

Appendix A: Transmission probability expressions for the Casimir conductances

Reference [7] finds the following expressions for the conductances in Casimir's equation [(8), (9)] relating two currents in a four-pole conductor to the chemical potentials:

$$\alpha_{11} = (e^2/h)[(1 - R_{11})S - (T_{14} + T_{12})(T_{41} + T_{21})]/S, \quad (\text{A1})$$

$$\alpha_{12} = (e^2/h)(T_{12}T_{34} - T_{14}T_{32})/S, \quad (\text{A2})$$

$$\alpha_{21} = (e^2/h)(T_{21}T_{43} - T_{23}T_{41})/S, \quad (\text{A3})$$

$$\alpha_{22} = (e^2/h)[(1 - R_{22})S - (T_{21} + T_{23})(T_{32} + T_{12})]/S, \quad (\text{A4})$$

where

$$S = T_{12} + T_{14} + T_{32} + T_{34} = T_{21} + T_{41} + T_{23} + T_{43}. \quad (\text{A5})$$

From Equation (5) it follows that the diagonal coefficients are symmetric in the flux, and the off-diagonal elements obey $\alpha_{12}(\Phi) = \alpha_{21}(-\Phi)$. The β conductances describing the situation where a current I_1 flows from terminal 1 to terminal 4 and a current I_2 flows from terminal 3 to terminal 2 are obtained by the substitution $3 \rightarrow 4, 4 \rightarrow 2, 2 \rightarrow 3$ in Equations (A1)–(A5). The γ conductances describing the situation where a current I_1 flows from terminal 1 to terminal 2 and a current I_2 flows from terminal 4 to terminal 3 are obtained by the substitution $3 \rightarrow 2, 2 \rightarrow 4, 4 \rightarrow 3$ in Equations (A1)–(A5).

Appendix B: Four-terminal interpretation of the conductance formulae of Azbel [25] and Büttiker et al. [21]

A four-terminal interpretation of the results of [21] and [25] is explicit in the work of Sivan and Imry [34]. Below we show that such an interpretation is compatible with the reciprocity theorem, Equation (1), in the absence of a magnetic field, but that it contradicts the reciprocity theorem, Equation (3), in the presence of a magnetic field. To this extent we picture, as in Figure 4, two contacts separated by a tunnel barrier from the perfect leads. We introduce the total transmission and reflection probabilities into channel i for carriers incident from the left,

$$R_i = \sum_{j=1}^{j=N} R_{21,ij}, \quad T_i = \sum_{j=1}^{j=N} T_{21,ij}, \quad (\text{B1})$$

and channel i for carriers incident from the right,

$$R'_i = \sum_{j=1}^{j=N} R_{12,ij}, \quad T'_i = \sum_{j=1}^{j=N} T_{12,ij}. \quad (\text{B2})$$

In these papers the measured voltage is determined by the piled-up densities. The density in channel i on the left is proportional to $v_i^{-1} T_i$ due to transmitted carriers from reservoir 1. It is now assumed that the piled-up densities determine the flow of current from the conductor to the measurement probe. The current from all of the N channels to reservoir 4 is taken to be proportional to

$$T_{41} = T_{14} = \epsilon \sum_{i=1}^{i=N} v_i^{-1} T_i, \quad (\text{B3})$$

with a matrix element ϵ coupling *all* the channels of the conductor *equally* to those of the measurement lead. Note that in this discussion coupling of the conductor to the measurement leads is described by a single parameter ϵ . Similar considerations give

$$T_{32} = T_{23} = \epsilon \sum_{i=1}^{i=N} v_i^{-1} T'_i, \quad (\text{B4})$$

$$T_{31} = T_{13} = \epsilon \sum_{i=1}^{i=N} v_i^{-1} (1 + R_i), \quad (\text{B5})$$

$$T_{42} = T_{24} = \epsilon \sum_{i=1}^{i=N} v_i^{-1} (1 + R_i'). \quad (\text{B6})$$

Furthermore, $T_{34} = T_{43} = 0$ to order ϵ . To lowest order in ϵ , the transmission from reservoirs 1 to 2 is unaffected by the probes, $T = T_{21} = T_{12}$. T has the symmetry of the two-terminal transmission coefficient, Equation (16). Using these results in Equation (7) or Equation (8) yields

$$\mathcal{R}_{12,34} = \left(\frac{h}{e^2}\right) \frac{1}{T} \frac{\sum \frac{1}{v_i} (1 + R_i - T_i)}{2 \sum \frac{1}{v_i}}. \quad (\text{B7})$$

Equation (B7) is the main result of [21] and [25]. In the absence of a magnetic field, the transmission probabilities given above have the required reciprocity symmetry for the transmission probabilities associated with a four-terminal conductor. However, in the presence of a flux Φ , the transmission probabilities given above do not obey [Equation (5)], $T_{ij}(\Phi) = T_{ji}(-\Phi)$. From Equation (B3) we find

$$T_{41}(\Phi) - T_{14}(-\Phi) = \epsilon \sum_{i=1}^{i=N} v_i^{-1} [T_i(\Phi) - T_i(-\Phi)], \quad (\text{B8})$$

which is in general not zero, since $T_i(\Phi) \neq T_i(-\Phi)$. As pointed out in Section 5, the results of Azbel [25] and Büttiker et al. [21] relate a spatially averaged local electric potential difference ΔU to a resistance, not a genuine chemical potential difference $\Delta\mu$. From this viewpoint it is not surprising that Equation (B8) is not symmetric under magnetic field reversal, as pointed out in [24]. The reciprocity symmetry (3) is a consequence of the relation between currents and chemical potentials, but says nothing about the relation of electric potentials and currents.

We conclude this appendix with an estimate of the magnetic field asymmetry [24] predicted by Equation (B7). We consider a piece of conductor of length l_ϕ (the phase-breaking length) and with an elastic scattering length l_e . The ensemble-averaged resistance is $\langle \mathcal{R}_\phi \rangle = \langle (h/e^2) T^{-1} \rangle = (h/e^2)(l_\phi/Nl_e)$, where N is the number of quantum channels. Using the argument of Lee [71] that the reflection coefficients fluctuate independently yields $\langle [\mathcal{R}_{12,34}(\Phi) - \mathcal{R}_{12,34}(-\Phi)]^2 \rangle \cong (h/e^2)^2 (l_e/l_\phi)^2$. Since $l_e \ll l_\phi$, this is much smaller than the experimentally observed asymmetry [27, 41] which is of the order of $(h/e^2)^2 \mathcal{R}_\phi^4$, independent of the separation of the voltage probes [41]. Equation (B7) predicts a small asymmetry because it is determined by spatially averaged voltages and because the physics of the contacts is neglected.

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