DISCUSSION OF PROBABILITY RELATIONS BETWEEN SEPARATED SYSTEMS

By E. SCHRÖDINGER

[Communicated by Mr M. Born]

[*Received* 14 August, *read* 28 October 1935]

1. When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or ψ-functions) have become entangled. To disentangle them we must gather further information by experiment, although we knew as much as anybody could possibly know about all that happened. Of either system, taken separately, all previous knowledge may be entirely lost, leaving us but one privilege: to restrict the experiments to one only of the two systems. After re-establishing one representative by observation, the other one can be inferred simultaneously. In what follows the whole of this procedure will be called *the disentanglement*. Its sinister importance is due to its being involved in every measuring process and therefore forming the basis of the quantum theory of measurement, threatening us thereby with at least a *regressus in infinitum*, since it will be noticed that the procedure itself involves measurement.

Another way of expressing the peculiar situation is: the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of all its *parts*, even though they may be entirely separated and therefore virtually capable of being "best possibly known", i.e. of possessing, each of them, a representative of its own. The lack of knowledge is by no means due to the interaction being insufficiently known—at least not in the way that it could possibly be known more completely—it is due to the interaction itself.

Attention has recently* been called to the obvious but very disconcerting fact that even though we restrict the disentangling measurements to *one* system, the representative obtained for the *other* system is by no means independent of the particular choice of observations which we select for that purpose and which by

* A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* 47 (1935), 777.

the way are *entirely* arbitrary. It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it. This paper does not aim at a solution of the paradox, it rather adds to it, if possible. A hint as regards the presumed obstacle will be found at the end.

2. To begin with I wish to establish a simple theorem, which makes it very obvious that the phenomenon in question is a quite general one; that it is the rule and not the exception. The representative arrived at for *one* system depends on the *programme* of observations to be taken with the other one. It is necessary to envisage the dependence on the *programme*. For since one device only can be carried out in every individual case and since, moreover, we cannot tell the result (because after all we are not actually experimenting, but sitting at our desk), there seems to be a certain liberty for presuming that perhaps, after all, there is always or at least in most cases a result possible, which is also possible when other devices are followed, and that perhaps it is this that actually would turn up.

Let x and y stand for all the coordinates of the first and second systems respectively and $\Psi(x, y)$ for the normalized representative of the state of the composed system, when the two have separated again, after the interaction has taken place. What constitutes the entanglement is that Ψ is not a product of a function of x and a function of y. Now suppose that we perform on the second system certain observations in consequence of which its representative, at the moment in which disentanglement is reached, is sure to turn up as one out of the known complete set of normalized orthogonal functions $f_n(y)$. Then, provided that the variables which we have measured all commute, we have to develop $\Psi(x, y)$ into a series with respect to the f_n,

$$\Psi(x, y) = \sum_n c_n g_n(x) f_n(y), \tag{1}$$

in order to come to know the representative of the other system. When the readings on the y-system point to $f_k(y)$, we have to adopt $g_k(x)$ as the representative of the x-system. The c_k have been introduced in order to assume that the g_k are normalized, i.e. that

$$\int g_k^*(x)\, g_k(x)\, dx = 1. \tag{2}$$

Of course $|c_k|^2$ is the probability of that particular case occurring. The equations

$$c_k g_k(x) = \int f_k^*(y)\, \Psi(x, y)\, dy \tag{3}$$

together with (2) determine the c's and the g's, apart from an irrelevant phase-factor in every g and its reciprocal in the corresponding c *and* apart from the possible indeterminateness of g, should the integral for some values of k vanish identically in x.

There is no reason for the g_k to be orthogonal to each other. Let us ask *when* they are, i.e. *how* must the f_k be chosen for that purpose? The condition evidently is

$$c_k^* c_l \delta_{kl} = \int dx \int dy \int dy' f_k(y') \Psi^*(x, y') f_l^*(y) \Psi(x, y). \tag{4}$$

This amounts to saying that, for every k, the function

$$u_k(y) = \int dx \int dy' f_k(y') \Psi^*(x, y') \Psi(x, y) \tag{5}$$

is to be orthogonal to all the $f_l(y)$, with the possible exception of $f_k(y)$. Hence $u_k(y)$ must be a numerical multiple of $f_k(y)$. From (4), with $l = k$, it is seen that the numerical multiplier is $|c_k|^2$. We have therefore

$$|c_k|^2 f_k(y) = \int dx \int dy' f_k(y') \Psi^*(x, y') \Psi(x, y). \tag{6}$$

Introducing the function

$$K(y, y') = \int dx \, \Psi^*(x, y') \Psi(x, y), \tag{7}$$

which has Hermitian symmetry, we see from (6) that the reciprocals of the $|c_k|^2$ and the functions $f_k(y)$ are required to be the eigenvalues and a system of eigenfunctions respectively of the homogeneous integral equation

$$f(y) = \lambda \int K(y, y') f(y') \, dy'. \tag{8}$$

Provided that the integral in (7) converges, so that K is defined, a complete solution of (8) exists. (It is convenient for our purposes, in order to be concerned with *complete* sets only, to include the functions, orthogonal to K, as eigenfunctions belonging to $\lambda = \infty$, at variance with the custom of mathematicians.) By using this set for the development (1) one easily satisfies oneself that all requirements are fulfilled, in particular that the λ_k^{-1} *are* all non-negative and that their sum *is* unity.

The general case is evidently that all the λ_k^{-1} are different from one another, except maybe for an arbitrary set of them vanishing. Then the *relevant* $f_k(y)$ are uniquely determined and so are the $g_k(x)$. Hence there is always *one* and as a rule only one development of $\Psi(x, y)$ of the type which might suitably be called "biorthogonal†"

Whenever (and of course only when) the eigenfunctions of a programme to be carried out on the y-system include the relevant functions $f_k(y)$, or the eigenfunctions properly speaking of (8), the programme will lead to the biorthogonal development and imply the relevant $g_k(x)$ as the other set. Now if for an arbi-

† The whole mathematical treatment is familiar to mathematicians in dealing with an "unsymmetrical kernel" $\Psi(x, y)$. See Courant-Hilbert, *Methoden der mathematischen Physik*, 2nd edition, p. 134.

trarily fixed programme of measurements on the y-system the representative arrived at for the x-system was the same *in all individual cases*, the same $g_k(x)$ would have to turn up (and even with the same probabilities) as in the biorthogonal development; for in two infinite series of repetitions *ab ovo* of one and of the other programme respectively every possible result occurs according to its due probability. Hence the relevant functions $g_k(x)$ would have to be implied whatever programme is carried out. But since, of course, they also determine the biorthogonal development uniquely and thereby require the relevant $f_k(y)$ as the other set, these would have to be included in the eigenfunctions of every programme which cannot be, since the latter are, by principle, an entirely arbitrary complete orthogonal set. Hence the non-invariance is proved*.

There must, of course, be cases in which the biorthogonal development refers to a continuous variable (or set of commuting variables), an integral replacing the series (1); and also mixed cases. In our present treatment they would be indicated by the integral (7) diverging and would therefore require a separate treatment, on which I shall not enter here.

The biorthogonal development is the one to give us true insight into the entanglement. If there are no coincidences among the $|c_k|^2$ (excluding also the case, that more than one of them vanish) the relevant f_k's form a well-determined and complete set and so do the g_k's. Then one can say that the entanglement consists in that one and only one observable (or set of commuting observables) of one system is uniquely determined by a definite observable (or set of commuting observables) of the other system. This is the general case. We shall now turn to the opposite extreme, which is the Einstein-Podolsky-Rosen case. It could be characterized by *all* the $|c_k|^2$ being equal and *all* possible developments being biorthogonal. *Every* observable (or set, etc.) of one system is determined by an observable (or set, etc.) of the other one. But the mere fact, that the equality of the $|c_k|^2$ prevents their sum from being normalized to unity, shows us that very improper representatives (in fact much more so than Dirac's δ, δ', δ'', ...) are involved in this case, making it advisable to deal with it on slightly different lines.

3. For simplicity's sake we suppose each of the two systems to have one degree of freedom only. Let the q-numbers x_1, p_1 and x_2, p_2 denote coordinate and momentum of the first and of the second system respectively. The existence of further degrees of freedom would not affect the considerations of this section except for slight alterations in the wording; but for section 4 to hold it would have to be assumed, that within each of the two systems the degree of freedom which we investigate has its Hamiltonian separated from the rest.

* In order to adapt this proof to the case when the biorthogonal development is *not* unique, just replace *the* biorthogonal development by a particular one, on which you fix your attention.

The two systems are of course supposed not to interact with each other. The entanglement is to be such that the two *commuting* observables

$$x = x_1 - x_2, \quad p = p_1 + p_2, \tag{9}$$

which we choose to represent the state of the *composed* system, have definite numerical values, say x' and p' respectively, which we suppose to be known. The representative Ψ of the composed system is a function of the eigenvalues of x and p, which involves x' and p' as parameters and vanishes everywhere except in that point where the former are equal to the latter. It is not a δ-function though and can hardly be written explicitly. According to our assumptions Ψ must have the properties

$$x\Psi = x'\Psi \quad \text{and} \quad p\Psi = p'\Psi. \tag{10}$$

We shall use no others.

From (9) the variable x can be observed by observing x_1 and x_2 separately, because the latter commute. The difference of the observed values, x_1' and x_2' say, must be equal to x':

$$x_1' - x_2' = x'. \tag{11}$$

Hence x_1' can be predicted from x_2' and *vice versa*. Similarly

$$p_1' + p_2' = p', \tag{12}$$

so that the result of measuring p_1 serves to predict the result for p_2 and *vice versa*. But of course every *one* of the *four* observations in question, when actually performed, disentangles the systems, furnishing each of them with an independent representative of its own. A *second* observation, whatever it is and on whichever system it is executed, produces no further change in the representative of the *other* system.

Yet since I can predict *either* x_1' *or* p_1' without interfering with system No. 1 and since system No. 1, like a scholar in examination, cannot possibly know which of the two questions I am going to ask it first: it so seems that our scholar is prepared to give the right answer to the *first* question he is asked, *anyhow*. Therefore he must know both answers; which is an amazing knowledge, quite irrespective of the fact that after having given his first answer our scholar is invariably so disconcerted or tired out, that all the following answers are "wrong"

Thus far the results of the paper quoted above. Now I wish to point out that system No. 1 (say) has further knowledge. It does not only know these two answers but a vast number of others, and that with no mnemotechnical help whatsoever, at least with none that we know of.

Let us consider an Hermitian operator referring to the first system and given as a "well-ordered" analytic function of the observables x_1 and p_1:

$$F(x_1, p_1), \tag{13}$$

which we suppose not to contain the $\sqrt{-1}$ *explicitly*. It is an observable of system No. 1. We shall prove that its value is equal to the value of the following observable of system No. 2

$$F(x_2 + x', p' - p_2), \tag{14}$$

https://doi.org/10.1017/S0305004100013554 Published online by Cambridge University Press

so that the result of either observation can be predicted from the other one. That is not trivial, because the equations $x=x'$ and $p=p'$ do not hold, except in the form (10), that is to say they are not identities.

The proof will be produced, if we can show that the difference of the two operators, when applied to Ψ, gives zero:

$$\{F(x_2+x', p'-p_2)-F(x_1, p_1)\}\Psi=0. \tag{15}$$

Using (9), we may write this in the form

$$\{F(x_1+x'-x, p_1+p'-p)-F(x_1, p_1)\}\Psi=0. \tag{16}$$

To prove it we observe that any operator which *ends* on its right with a factor $x'-x$ or $p'-p$ reduces Ψ to zero, from (10). Additive terms of this type can therefore be dropped within the curved bracket. Now fix the attention on one of the power products in the minuendus. Its last factor, either $x_1+x'-x$ or $p_1+p'-p$, can be replaced by x_1 or p_1, as the case may be, and then this x_1 or p_1 commutes with the rest of the power product and can be removed to its extreme left. The second step consists in applying a similar treatment to the factor ($x_1+x'-x$ or $p_1+p'-p$, as the case may be), which has *now* become the last; but this one cannot safely be displaced to the *extreme* left but only to the second place, counting from the left. This procedure is continued until we are left with a power product which differs from the original one in that x_1 and p_1 have replaced $x_1+x'-x$ and $p_1+p'-p$ respectively and also that the order of factors has been reversed. But F, owing to its Hermiticity and to the further condition that it should not contain $\sqrt{-1}$ explicitly, must contain the "reversed" power product too. Hence after applying the same treatment to all of them, we are left with $F(x_1, p_1)$, which cancels with the subtrahendus, and the statement is proved.

If F contains the $\sqrt{-1}$ explicitly, we could replace it by $(x_1p_1-p_1x_1)/\hbar$. Then the prescription (14) would apply without corollary. It would turn the operator just mentioned into $(p_2x_2-x_2p_2)/\hbar$ which now can be replaced by $-\sqrt{-1}$. From this follows the corollary to prescription (14), that an explicit $\sqrt{-1}$ has to change sign.

By this theorem all observables are placed on the same footing. Our system, in its virgin state, must know the answers to all of them. One might presume that it avails itself at least of a suggestive mnemotechnical device, viz. that the answer prepared for the variable $F(x_1, p_1)$ is simply $F(x_1', p_1')$, if x_1' and p_1' are those prepared for x_1 and p_1 respectively. But this is not so. For consider, e.g., the *series* of observables

$$F(x_1, p_1, b)=\frac{1}{b}p_1^2+bx_1^2,$$

where b is to be a positive c-number parameter. With every value for b we are confronted with a new observable, to which an answer must be pending. Moreover the answer must be, irrespective of b, an odd integral multiple of $\hbar$ (though not necessarily independent of b). This shows plainly that all these answers cannot conform to the results which would be obtained by inserting into the expression the same pair of c-numbers, p_1' and x_1'; which, by the way, are simultaneously accessible to experiment in every individual case, one by direct observation, the other one by inference from an observation on the other system.

Our complete lack of insight into the relationship between the different answers in *one* system is all the more bewildering, since we have proved, on the other hand, that the one-to-one correspondence between the answers of the two systems necessarily extends to *all* pairs of observables whenever it holds for two of them.

4. If equations (10) are assumed to hold at time zero, the equations of motion determine what becomes of them as time proceeds. Let the Hamiltonian of the composed system be

$$H = H_1(x_1, p_1) + H_2(x_2, p_2) \tag{17}$$

and let it not contain the time explicitly. We shall use what Dirac calls a Heisenberg representation; then every variable at time t is a function of the variables at time zero, e.g.

$$x_{1t} = e^{itH_1/\hbar}\, x_1\, e^{-itH_1/\hbar}. \tag{18}$$

From prescription (14), including the corollary, we can find out what observable of system No. 2 is equivalent to x_{1t}; we call it $[x_{1t}]_2$,

$$[x_{1t}]_2 = e^{-itH_1(x_2+x',\, p'-p_2)/\hbar}\,(x_2+x')\,e^{itH_1(x_2+x',\, p'-p_2)/\hbar}. \tag{19}$$

This equation, by its form, indicates the observation on No. 2 *at time zero*, which would serve to predict the coordinate of No. 1 at time t. Solving two equations similar to (18) for x_2 and p_2, we get

$$x_2 = e^{-itH_2/\hbar}\, x_{2t}\, e^{itH_2/\hbar} \tag{20}$$

and similarly for p_2. Of course H_2 has now to be thought of as written with the arguments x_{2t} and p_{2t}; which does not affect its form, since it is a constant of the motion. With these expressions replacing x_2 and p_2 in (19), the exponentials with H_2 cancel in the interior, leaving just one in front and in the rear. So the final result is

$$[x_{1t}]_2 = e^{-itH_2/\hbar}\, e^{-itH_1/\hbar}\,(x_{2t}+x')\,e^{itH_1/\hbar}\,e^{itH_2/\hbar}, \tag{21}$$

where H_1 and H_2 are precisely the functions of equation (17), but written with the arguments

$$x_{2t}+x' \quad \text{and} \quad p'-p_{2t} \quad \text{for } H_1,$$
$$x_{2t} \quad \text{and} \quad p_{2t} \quad \text{for } H_2.$$

This rather complicated function of x_{2t} and p_{2t} is *that* observable of system No. 2 which is equivalent to x_{1t}. Though we have deduced it by means of a Heisenberg representation, the functional connection is of course exactly the same for what Dirac calls the Schrödinger representatives. That is to say, we can take x_{2t} and p_{2t} to have the general meaning of x_2 and p_2 of the preceding section. Regarded as operators they then do not involve the notion of time but work on a Ψ-function, which develops according to the wave equation. This consideration applies to every moment of time. It is therefore correct to say, that the variable which in No. 2 is equivalent to the coordinate in No. 1 undergoes a continuous unitary or contact transformation as time goes on. The transformation is of course the same for every observable, so that we need not write out the formulae for $[p_{1t}]_2$ or for an arbitrary $[F(x_{1t}, p_{1t})]_2$. It is noteworthy that the two exponentials of which the transformation is composed may not be amalgamated, because H_1 and H_2, considering the arguments with which they are written, do not in general commute.

All this is moderately trivial. But it is necessary to consider it lest one should believe that the antinomies could be solved by suggesting or proving that some of the observations must take a certain minimum time. Provided that they *relate* to a definite moment, this will not help us. It cannot be argued that, before the results are reached, the situation to which they refer has passed away. A prediction for time zero does not dissolve into nought as time goes on, but simply transforms into the prediction of another observable. And any desired observable can be predicted for time t by making a *suitable* observation at time zero on the *other* system.

When at time zero a certain observable of system No. 1, say x_1, is inferred from observing x_2, *I am forced* to assign to system No. 1 a representative that makes the observable x_1 precise and tells nothing about its canonically conjugate, *although* I safely infer that system No. 1 *does* know quite a definite (as opposed to a haphazard) answer for the canonical conjugate as well, the only difference being that I know the one while I am ignorant of the other. Now this paradoxical situation is not confined to time zero and could not, therefore, be avoided by my satisfying myself that the result of observing x_2 cannot be known before a certain time has elapsed. From the moment I come to know the result I should be faced with exactly the same situation, only the pair of canonically conjugate observables that is involved changes with time.

The paradox would be shaken, though, if an observation did not *relate* to a definite moment. But this would make the present interpretation of quantum mechanics meaningless, because at present the *objects* of its predictions are considered to be the results of measurements for definite moments of time.

My sincerest gratitude is due to Imperial Chemical Industries, Limited, to whose generosity I owe the leisure for carrying out these studies.

SUMMARY

The probability relations which can occur between two separated physical systems are discussed, on the assumption that their state is known by a representative in common. The *two families* of observables, relating to the first and to the second system respectively, are linked by at least *one match* between two definite members, one of either family. The word *match* is short for stating that the *values* of the two observables in question determine each other uniquely and therefore (since the actual labelling is irrelevant) can be taken to be *equal*. In general there is but one match, but there can be more. If, in addition to the first match, there is a second one between canonical conjugates of the first mates, then there are infinitely many matches, every function of the first canonical pair matching with the same function of the second canonical pair*. Thus there is a complete one-to-one correspondence between *those* two branches (of the two families of observables) which relate to the two degrees of freedom in question. If there *are* no others†, the one-to-one correspondence persists as time advances, but the observables of the first system (say) change their mates in the way that the latter, i.e. the observables of the second system, undergo a certain continuous contact-transformation.

* To make the earlier text conform to the present simplified wording, replace $x_2 + x'$ by P and $p' - p_2$ by X. Then X and P are canonical conjugates. The mating (x with P and p with X) has to be *cross-wise*, though.

† In fact it persists anyhow, but as a rule in a very much more complicated form.