On the Laws of Radiation.

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1. An attempt to obtain the law of partition of the radiation proceeding from a radiating body calls at the outset for a consideration of the partition of energy between the matter of which the radiating body is composed, and the ether by which it is surrounded. This question has been discussed by Lord Rayleigh* and by the present author.† Assuming that the ultimate state of equilibrium between the energies of matter and ether has been reached, the theorem of equipartition of energy enables us to determine the amount not only of the total energy of the ether, but also of the energy of each wave-length. It is found that at a temperature T, the energy per unit volume of radiation consisting of waves of wave-lengths between λ and $\lambda + d\lambda$ is

$$8\pi \mathrm{RT}\lambda^{-4}d\lambda.$$

It is obvious that this law, according to which the energy tends to run entirely into waves of infinitesimal wave-length, cannot be the true law of partition of the radiant energy which actually occurs in nature. The law is obtained from the supposition that a state of statistical equilibrium has been arrived at between the energies of different wave-lengths and that of matter; the inference to be drawn from the failure of this law to represent natural radiation is that in natural radiation such a state of equilibrium does not obtain. An analogous situation presents itself in the theory of gases. According to the theorem of equipartition of energy, the energy of a gas will ultimately be almost entirely absorbed by the modes of internal vibration of its molecules, whereas it is known that in nature only a very small fraction of the energy is possessed by these internal vibrations. Thus we are led to suppose that there is not a state of equilibrium between the internal vibrations of the molecules and their energy of translation; we find that the transfer of energy from the translational to the vibrational degrees of freedom is so slow that the latter degrees never acquire their full share of energy, as given by the theorem

* "On the Dynamical Theory of Gases and Radiation," 'Nature,' May 18, July 13, 1905.

† "On the Partition of Energy between Matter and Ether," 'Phil. Mag.' [6], vol. 10, p. 97.

of equipartition, the energy of these vibrations being dissipated away as rapidly as it is received from the translational energy of the molecules. A similar explanation suggests itself in the case of the partition of radiant energy. In the present paper an attempt is made to show that such an explanation is in accordance with the observed facts. The radiant energy acquired by the ether, whether of small or of great wave-length, is reabsorbed by other bodies or is radiated away into space, in such a way that the partition of the energy actually present in the ether at any instant is entirely different from that predicted by the law of equipartition.

On this view, the true laws of radiation can be obtained only through a study of the processes of transfer of energy from matter to ether. This in turn demands the use of hypotheses or assumptions as to the structure of matter and the mechanism by which radiation is produced. The view of the genesis of radiation which will be taken in the present paper is one which has already been put forward and developed to a considerable extent by Lorentz.*

2. The radiation from all solid bodies shows the same general characteristics, and according to this view, the common basis of these characteristics is the presence of free electrons in the source of radiation. As the electrons describe curved paths, they yield up energy to the ether, and the part of this energy which escapes reabsorption by matter figures as the energy of radiation represented in the continuous spectrum of a solid.

In addition to free electrons, there will also be present in the radiator a number of atoms, these also being charged with electricity. The atoms and complete molecules of the solid must be supposed to be so closely packed that there is not room for them to move past one another. Thus these bodies merely oscillate about their positions of equilibrium, while the electrons thread their way through the interstices. No matter what forces are at work between the electrons and the larger masses, or how closely the latter are packed, the law of distribution of velocities of the molecules, of the atoms, and of the electrons, will be Maxwell's law.† From this law, it follows that the velocities of the atoms and molecules will, on account of their greater mass, be very much smaller on the average than the velocities of the free electrons. Thus the atoms or molecules may, with considerable closeness of approximation, be regarded simply as fixed obstacles, against which the electrons impinge. The emitted radiation being regarded as the result solely of the motion of electric charges, it follows that the slow motion of complete

* "On the Emission and Absorption by Metals of Rays of Great Wave-length," Akad. van Wetenschappen, Amsterdam, April 24, 1903.

† *Cf.* the author's "Dynamical Theory of Gases," § 86.

atoms or molecules will contribute but little to the total radiation, this radiation proceeding almost entirely from the more rapid motions of the free electrons as they thread their way through the solid.

As the electrons move, that part of their energy which is yielded up to the ether, assumes the form of radiant energy, and travels through the ether in all directions with a uniform velocity V, the velocity of light, except in so far as this radiant energy is obstructed or reabsorbed by matter. Let us suppose the boundary of the radiator to be a semi-infinite plane. Then if κ is a coefficient of extinction, the energy generated at a point distant r from some specified small area on the boundary, will, by the time it crosses the boundary through this small area, be reduced in the ratio $e^{-\kappa r}$. Supposing energy to be yielded up to the ether at a uniform rate G per unit volume per unit time throughout the radiator, the stream of energy crossing unit area of the boundary

$$= \int_0^\infty \frac{G}{4\pi r^2} e^{-\kappa r^2} \pi r^2 \, dr = \frac{G}{4\kappa},$$

a definite finite quantity. The contribution to the total stream of radiation from large values of r is infinitesimal, so that we may regard the stream of energy which crosses the boundary at any point as proceeding only from those parts of the radiator which are in the immediate neighbourhood of the point. Thus the shape and size of the radiator do not influence the stream of radiation issuing from a point on its surface; it is only the structure of the surface-layers at the point which is of importance.

3. Let us suppose that at any point of the surface the energy of the issuing radiation, of which the wave-length lies between λ and $\lambda + d\lambda$ is

$$\phi(\lambda, T)\, d\lambda \qquad \text{(i)}$$

per unit volume, T being the temperature of the radiator. Our problem is to discuss the form of the function ϕ.

In addition to depending on λ and T, the function ϕ will involve the following quantities:—

V, the velocity of light;

e, the charge of an electron;

m, the mass of an electron;

R, the constant of the theory of gases, this being such that the mean kinetic energy of an electron is $\frac{3}{2}$RT;

K, the inductive capacity of the ether, measured in whatever units are in use;

$k_1, k_2, k_3, \ldots$, quantities specifying the structure of the radiating solid, *e.g.*, the number of free electrons per unit volume, the shape, size, mass, etc., of the atoms and molecules.

Thus the law of radiation can be expressed more completely by

$$\phi\,(\lambda, \mathrm{T}, \mathrm{V}, e, m, \mathrm{R}, \mathrm{K}, k_1, k_2, \ldots)\, d\lambda. \tag{ii}$$

By solving the equations

$$\frac{\partial}{\partial k_1}\int \phi\, d\lambda = 0, \qquad \frac{\partial}{\partial k_2}\int \phi\, d\lambda = 0, \text{ etc.}, \tag{iii}$$

we obtain the values of k_1, k_2...for which the function $\int \phi\, d\lambda$ possesses its maximum value. We therefore obtain a knowledge of the properties of the solid body, which is such that the total radiation at a given temperature is a maximum. Let us refer to this body as the "radiator of maximum efficiency" for the temperature in question, and let us denote the value of ϕ for this "radiator of maximum efficiency" by ϕ_m. If we eliminate k_1, k_2... from equations (ii) and (iii) we obtain ϕ_m as a function of

$$\lambda, \mathrm{T}, \mathrm{V}, e, m, \mathrm{R}, \text{ and } \mathrm{K}. \tag{iv}$$

4. In terms of the units, L of length, M of mass, t of time, K of inductive capacity, and T of degrees of temperature, the physical dimensions of these seven quantities are as follows :—

λ	is of dimensions	....	L
T	"	....	T
V	"	....	$\mathrm{L}t^{-1}$
e	"	....	$\mathrm{L}^{\frac{3}{2}}\mathrm{M}^{\frac{1}{2}}t^{-1}\mathrm{K}^{\frac{1}{2}}$
m	"	....	M
R	"	....	$\mathrm{LM}t^{-2}\mathrm{T}^{-1}$
K	"	....	K

Here are seven quantities and five independent physical units. It must, therefore, be possible to combine the seven quantities in two independent ways so as to form a mere number. We may take as two independent expressions formed from these seven quantities, so as to have the dimensions of a number,

$$c_1 \equiv \mathrm{RT}m^{-1}\mathrm{V}^{-2}, \qquad c_2 \equiv \lambda\mathrm{RTK}e^{-2}.$$

Any other pure number which can be formed from these seven quantities must be of the form $f(c_1, c_2)$.

The physical dimensions of the function ϕ_m are those of energy per unit volume per unit wave-length, hence ϕ_m is of dimensions $\mathrm{L}^{-3}\mathrm{M}t^{-2}$. These dimensions are those of $\lambda^{-4}\mathrm{RT}$. The ratio of ϕ_m to this quantity is, therefore, a pure number, and from this it follows that it must be possible to express ϕ_m in the form

$$\phi_m = \lambda^{-4}\mathrm{RT}f(c_1, c_2). \tag{v}$$

5. There is an obvious physical interpretation of the number c_1. The mean kinetic energy of a free electron at temperature T is $\frac{3}{2}$RT, so that the value of C^2, the mean square of its velocity, is $\frac{3}{2}RTm^{-1}$. Thus $\frac{3}{2}c_1$ is equal to C^2/V^2. At a temperature of 100° C., the value of C is 7×10^6 cm. per second,* while the value of V is 3×10^{10}. The value of c_1 is accordingly $3 \cdot 6 \times 10^{-8}$, a quantity sufficiently small to be neglected. On passing to the limit in which c_1 is put equal to zero, the function $f(c_1, c_2)$ either may or may not tend to a definite limit $f(0, c_2)$. For the present we shall assume such a limit to exist, without entering upon a discussion of the exact meaning of this assumption.

6. On this assumption, since the actual value of c_1 is very small, we find that ϕ_m may very approximately be expressed in the form

$$\phi_m = \lambda^{-4} RT f(c_2);$$

or, replacing c_2 by its value and dropping the universal constants R, T, K and e,

$$\phi_m = \lambda^{-4} T f(\lambda T).$$

Thus the law of radiation from the radiator of maximum efficiency for temperature T and wave-length λ is

$$\lambda^{-4} T f(\lambda T)\, d\lambda.$$

7. The radiator of maximum efficiency has been defined so as to have reference to a given temperature. It is a purely ideal body, and there is no evidence given as yet whether or not its properties can be obtained even approximately from actual matter. The values of k_1, k_2, ... given by equations (iii) are functions of T, so that even if a natural body approximates closely to the radiator of maximum efficiency at a given temperature, it cannot be expected to do so at all temperatures. At the same time it is possible for us to imagine a purely ideal radiator which shall possess the property of being the radiator of maximum efficiency at all temperatures, the properties of this body changing in such a way that equations (iii) are satisfied at all temperatures. Let us agree for the present to speak of such a body as a "perfect radiator."

8. The law of radiation for a perfect radiator has been seen to be

$$\lambda^{-4} T f(\lambda T)\, d\lambda. \qquad \text{(vi)}$$

On integrating with respect to λ, we obtain at once that the total radiation is of the form σT^4, where $\sigma = \int_0^\infty x^{-4} f(x)\, dx$. This is the expression of Stefan's law. Let the wave-length at which the energy per unit wave-

* *Cf.* "The Dynamical Theory of Gases," p. 209.

length, *i.e.*, the coefficient of $d\lambda$ in expression (vi), is a maximum, be denoted by λ_{max}; then we find at once that $\lambda_{max}T$ is a root of

$$\frac{\partial}{\partial x}\{x^{-4}f(x)\} = 0,$$

so that we have the relation

$$\lambda_{max}T = a, \qquad \text{(vii)}$$

where a is a constant. This is the mathematical expression of Wien's displacement-law. Thus we see that the two laws, which are usually obtained by thermodynamical arguments, can be obtained simply by an argument from physical dimensions, coupled with the hypothesis that the source of radiation is the motion of electrical charges.

9. The argument from physical dimensions can, however, be used in a second way: we can obtain by its help a rough numerical estimate of some of the quantities concerned. For instance the constant σ of Stefan's law is a function only of V, e, m, R and K, and is of dimensions $L^{-2}Mt^{-2}T^{-4}$. Now the only way in which the quantities V, e, m, R, and K can be combined so as to form a quantity of the dimensions of σ is through an expression of the form $e^{-6}R^4K^3$. Hence, by using an argument which I have explained in another place,* it can be shown that σ must be equal, as regards order of magnitude, to $e^{-6}R^4K^3$.

Similarly it can be shown that the constant a of Wien's law must be equal, as regards order of magnitude, to $e^2R^{-1}K^{-1}$.

The values of σ and a have been obtained experimentally.† In C.G.S. centigrade units, the value of σ obtained from Kurlbaum's experiments is $5{\cdot}32 \times 10^{-5}$, while the value of a obtained by Lummer and Pringsheim is 0·294. Let us use these values in conjunction with the approximate theoretical values already obtained, to deduce the value of e, the charge on the particles by which radiation is generated. On taking the value‡ $R = 9{\cdot}3 \times 10^{-17}$ we find from the approximate equality

$$e^{-6}R^4K^3 = 5{\cdot}32 \times 10^{-5},$$

the value

$$eK^{-\frac{1}{2}} = 1{\cdot}8 \times 10^{-10},$$

while from the second relation,

$$e^2R^{-1}K^{-1} = 0{\cdot}294,$$

* "On the Vibrations and Stability of a Gravitating Planet," 'Phil. Trans.,' A, vol. 201, p. 158.

† An account of experimental determinations of these constants is given by Lummer, 'Congrès de Physique, Paris (1901) Rapports,' vol. 2, pp. 92—96.

‡ *Cf.* the author's 'Dynamical Theory of Gases,' p. 113.

we obtain $eK^{-\frac{1}{2}} = 51 \times 10^{-10}$. The value of $eK^{-\frac{1}{2}}$ obtained experimentally,* is 3×10^{-10}. The difference between this and the values obtained above is not greater than may fairly be ascribed to the roughness of the method used. For example, the calculations would have been the same if e had been measured in "rational" electric units instead of those in common use, but we should then have had an experimental value equal to 4π times that mentioned above, say $eK^{-\frac{1}{2}} = 38 \times 10^{-10}$. The comparative agreement between the theoretically predicted value and the true value must, therefore, be regarded as evidence that we are on the right track in attempting to obtain the laws of radiation from the supposition that the radiation proceeds from the motion of electrons.

10. On this supposition, the law of radiation for waves of great wave-length has been determined by Lorentz. His analysis rests upon the assumption, which we have already made, that the velocity of the electrons is small compared with that of light, and he makes the further assumption that the motion of the electrons may be regarded as a series of free paths separated by instantaneous collisions. On these assumptions, he obtains as the emission from a radiator

$$8\pi ART\lambda^{-4}d\lambda, \qquad \text{(viii)}$$

where A is the coefficient of absorption, as determined by Drude's theory.† The second assumption, that of undisturbed free paths and instantaneous collisions, although doubtless reproducing in the main the essential physical properties of the motion, will not necessarily lead to results which are numerically exact. It may be found that the results obtained require to be modified by the introduction of a numerical multiplier, just as Clausius' formula for the mean free path in a gas requires to be multiplied by a numerical factor to allow for the varying velocities of the individual molecules. We may, however, infer from Lorentz's analysis that the radiation, for great values of λ, is accurately proportional to $T\lambda^{-4}d\lambda$.

We can also obtain some idea of the form assumed by the law of radiation when λ is very small. The rate at which radiation of short wave-length λ is produced by collisions of electrons will contain as its most important feature a factor of the form e^{-n}, where n is a large number, comparable with the ratio of the average duration of a collision to the period of vibration λ/V.‡ Thus the radiation when λ is very small will vanish in the same way as the exponential $e^{-\tau V/\lambda}$ where τ is comparable with the duration of a collision.

* J. J. Thomson, 'Phil. Mag.' [6], vol. 5, p. 335.

† Drude's 'Annalen,' vol. 1, p. 576.

‡ *Cf.* "On the Application of Statistical Mechanics to the General Dynamics of Matter and Ether," 'Roy. Soc. Proc.," vol. 76, p. 296, § 13.

On comparison with the form of the general law of radiation, as given by expression (vi), it is seen that this factor must be of the form $e^{-c/\lambda T}$, and it is worthy of notice that every empirical law of radiation reduces when λ is very small to a form in which a factor of this type is the factor of preponderating importance.

11. To summarise the information which has been obtained, we may say that:—

(1) The law of radiation from a perfect radiator is of the form

$$\lambda^{-4} T f(\lambda T)\, d\lambda,$$

so that Stefan's law and Wien's displacement law are accurately obeyed by the radiation from this ideal radiator.

(2) For large values of λT, the form of the function $f(\lambda T)$ approximates to a constant, a result due to Lorentz.

(3) For small values of λT, the form of the function $f(\lambda T)$ is such that it decreases very rapidly as λ decreases, finally vanishing in the same way as the function $e^{-c/\lambda T}$.

A discussion of the relation between the radiation from our ideal "perfect radiator" and that from actual bodies, may appropriately be reserved for a later paper.